Renormalization group: General models with N-component fields

We study now more general models with an N-component field (or order parameter), from the viewpoint of the renormalization group (RG). Indeed, one can find interesting physical systems for which the Hamiltonian does not have the $O(N)$ orthogonal symmetry of the models considered so far.

A first family of such models is characterized by the presence of several independent correlation lengths. This happens typically when the quadratic part of the Hamiltonian involves several unrelated parameters.

Generically, the different correlation lengths then diverge for different values of the temperature. The components of the fields that are not critical decouple and can be ignored in the study of the asymptotic large-distance behaviour (in the sense of the field integral, they can be integrated out).

One can thus restrict the study of RG properties to models that, like the $O(N)$ model, have only one correlation length in the disordered phase.

Models with only one correlation length. The generic Hamiltonians that generate only one correlation length are invariant under some symmetry group G acting on the field, subgroup of the $O(N)$ group, which admits only one quadratic invariant. Moreover, the field must transform under an irreducible representation of the group G.

As a consequence, the two-point correlation function in the disordered phase is proportional to the identity matrix in component space. Denoting now by σ_α the N components of the field, we can express this condition as

$$\langle \sigma_\alpha(x)\sigma_\beta(y)\rangle = \frac{1}{N}\delta_{\alpha\beta}\sum_{\gamma=1}^{N}\langle \sigma_\gamma(x)\sigma_\gamma(y)\rangle\,. \tag{55}$$

Moreover, we assume that the group G contains the reflection group $\mathbb{Z}_2$, $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$ as a subgroup and admits several, linearly independent, quartic invariant monomials in σ, as the example of the cubic anisotropy will illustrate.

For this class of models, the effective Hamiltonians thus have the same quadratic terms as the $O(N)$ symmetric Hamiltonian, but differ by the quartic contributions: they contain several independent terms of $\int \mathrm{d}^d x\, \sigma^4(x)$ type, one of them always being $O(N)$ symmetric:

$$\mathcal{H}(\boldsymbol{\sigma}) = \int \mathrm{d}^d x \left[\tfrac{1}{2} \left(\nabla_x \boldsymbol{\sigma}(x) \right)^2 + \tfrac{1}{2} r \boldsymbol{\sigma}^2(x) + \frac{1}{4!} \sum_a g_a V_a \left(\sigma(x) \right) \right],$$

where the monomials $V_a(\sigma)$ are quartic in the field $\boldsymbol{\sigma}$,

$$V_a(\lambda\sigma) = \lambda^4 V_a(\sigma),$$

and linearly independent.

An example: A model with cubic anisotropy

As an example of such more general models, we first examine a model that has only two σ^4 terms but still has interesting physics properties.

We consider an N-component field σ_α, $\alpha = 1, \ldots, N$ and a Hamiltonian invariant under the cubic group, the finite group of transformations generated by

$$\sigma_\alpha \mapsto -\sigma_\alpha \,, \quad \sigma_\alpha \leftrightarrow \sigma_\beta \quad \text{for all } \alpha \text{ and } \beta.$$

The model has been proposed to describe the critical properties of classical spin systems in which the interactions are modulated by an underlying cubic lattice.

The cubic symmetry group admits a unique quadratic invariant but two independent quartic invariants,

$$\left(\sum_\alpha \sigma_\alpha^2\right)^2, \quad \sum_\alpha \sigma_\alpha^4.$$

A symmetric critical Hamiltonian in continuum space, truncated at order σ^4 as justified by the analysis of relevant operators at the Gaussian fixed point near dimension 4, has thus the general form

$$\mathcal{H}_{\mathrm{c}}(\sigma) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \sum_\alpha \left[\nabla \sigma_\alpha(x)\right]^2 + \text{ higher derivatives} \right.$$
$$\left. + \tfrac{1}{2} r_{\mathrm{c}}(g,h) \sum_\alpha \sigma_\alpha^2(x) + \frac{g}{24} \Bigl(\sum_\alpha \sigma_\alpha^2(x) \Bigr)^2 + \frac{h}{24} \sum_\alpha \sigma_\alpha^4(x) \right\}.$$

Since the model has a unique quadratic invariant, the condition (55) is satisfied, the two-point function in the disordered phase is proportional to the identity matrix and r_{c} is determined by the condition $\tilde{\Gamma}^{(2)}(p=0)=0$.

The appearance of two quartic terms implies that, on the critical surface, the RG in dimension $d = 4 - \varepsilon$ now involves two parameters g, h.

Positivity constraints and the order of the transition. The two constants g, h must satisfy two conditions, $g + h \geq 0$, $Ng + h \geq 0$, to ensure that the Hamiltonian is bounded from below for $\sigma \to \infty$ and, thus, that the transition is second order (if the quartic Hamiltonian is not bounded higher powers of the field become important).

The first condition is obtained by choosing all σ_α but one to vanish, the second by taking them all equal.

These conditions imply, in particular, that if the RG flow leads to parameters $g(\lambda), h(\lambda)$ outside this domain, the terms of higher degree in σ in the expansion of the thermodynamic potential, *a priori* thought to be negligible, become important.

Therefore, the transition, in contradiction with the predictions of the quasi-Gaussian or mean field approximation, is generically weak first order.

Effective couplings: RG equations

The effective constants $g(\lambda)$ and $h(\lambda)$ at scale λ satisfy flow equations of the general form

$$\lambda \frac{\mathrm{d}g}{\mathrm{d}\lambda} = -\beta_g\big(g(\lambda), h(\lambda)\big),$$
$$\lambda \frac{\mathrm{d}h}{\mathrm{d}\lambda} = -\beta_h\big(g(\lambda), h(\lambda)\big).$$

A one-loop calculation, analogous to the calculation with only one parameter, determines the two β-functions at leading order for $g, h = O(\varepsilon)$, $\varepsilon \to 0$:

$$\begin{cases} \beta_g(g,h) = -\varepsilon g + \dfrac{1}{8\pi^2}\left(\dfrac{N+8}{6}g^2 + gh\right), \\ \beta_h(g,h) = -\varepsilon h + \dfrac{1}{8\pi^2}\left(2gh + \dfrac{3}{2}h^2\right). \end{cases} \tag{56}$$

One has to study the flow of the parameters g and h as a function of the initial conditions and the dilatation parameter λ.

Fixed points

We first look for fixed points for $\varepsilon = 4 - d > 0$ and discusses their stability for $\lambda \to \infty$ as a function of the integer N.

An interesting issue is the symmetry of the various fixed point Hamiltonians, in particular, the one corresponding to the stable fixed point.

The equations $\beta_g = \beta_h = 0$ both factorize into two linear equations. Combining them in the four possible ways, one finds:

(i) the Gaussian fixed point

$$g = h = 0\,;$$

(ii) the decoupled fixed point

$$g = 0\,, \quad h = \tfrac{16}{3}\pi^2\varepsilon\,,$$

which corresponds to N identical and decoupled copies of a model with a $\mathbb{Z}_2$ reflection symmetry (Ising model universality class);

(iii) the isotropic fixed point

$$h = 0\,, \quad g = \frac{48\pi^2}{N+8}\varepsilon\,,$$

which corresponds to a model with $O(N)$ symmetry, a symmetry larger than the cubic symmetry of the initial Hamiltonian (an example of emergent symmetry);

(iv) finally, the fixed point,

$$g = \frac{16\pi^2\varepsilon}{N}\,, \quad h = \frac{16\pi^2(N-4)\varepsilon}{3N}\,,$$

which is new and is called the cubic fixed point.

All fixed points belong to the half-plane $g \geq 0$. Only the cubic fixed point for $N < 4$ is such that $h < 0$. However, for $N \geq 1$, it satisfies the positivity condition $g+h \geq 0$ (and thus also $Ng+h \geq 0$). Thus, all fixed points satisfy the positivity condition (64), in agreement with a general result.

Linearized flow and eigenvalues

The local stability properties of the four fixed points are determined by the eigenvalues of the matrix $\mathbf{L}$ of the partial derivatives, with respect to g and h, of the functions $-\beta_g, -\beta_h$. One finds

$$\mathbf{L} = -\begin{pmatrix} \frac{\partial \beta_g}{\partial g} & \frac{\partial \beta_g}{\partial h} \\ \frac{\partial \beta_h}{\partial g} & \frac{\partial \beta_h}{\partial h} \end{pmatrix} = \varepsilon \mathbf{1} - \frac{1}{8\pi^2} \begin{pmatrix} \frac{N+8}{3} g + h & g \\ 2h & 2g + 3h \end{pmatrix}.$$

For the four fixed points, the eigenvalues of the matrix $\mathbf{L}$ are

$$\begin{aligned}
\text{Gaussian fixed point:} \quad & \varepsilon\,, && \varepsilon\,, \\
\text{Decoupled (Ising-like) fixed point:} \quad & \frac{1}{3}\varepsilon\,, && -\varepsilon\,, \\
\text{Isotropic } O(N) \text{ fixed point:} \quad & \frac{N-4}{N+8}\varepsilon\,, && -\varepsilon\,, \\
\text{Cubic fixed point:} \quad & \frac{4-N}{3N}\varepsilon\,, && -\varepsilon\,.
\end{aligned}$$

All non-Gaussian fixed points have one eigenvalue $-\varepsilon$ and have, therefore, at least one stable direction.

The Gaussian fixed point is unstable in all directions.

The decoupled fixed point has, for all N, one direction of instability.

The isotropic fixed point illustrates a general result: the isotropic fixed point is stable for $N < N_{\rm c}$ with $N_{\rm c} = 4 + O(\varepsilon)$.

The cubic fixed point is stable only if $N > N_{\rm c}$. At $N = N_{\rm c}$ the two last fixed points merge and then exchange roles.

RG induced asymptotic symmetry. For $N < N_{\rm c}$, the stable fixed point has an $O(N)$ symmetry. The form of the asymptotic correlation functions in the critical domain, of the singularities of thermodynamic quantities at $T_{\rm c}$, thus reflects more symmetry than in the initial microscopic model.

We have already met an analogous phenomenon: the cubic symmetry of the lattice leads to a continuous $O(d)$ spatial symmetry at large distance or in the critical domain.

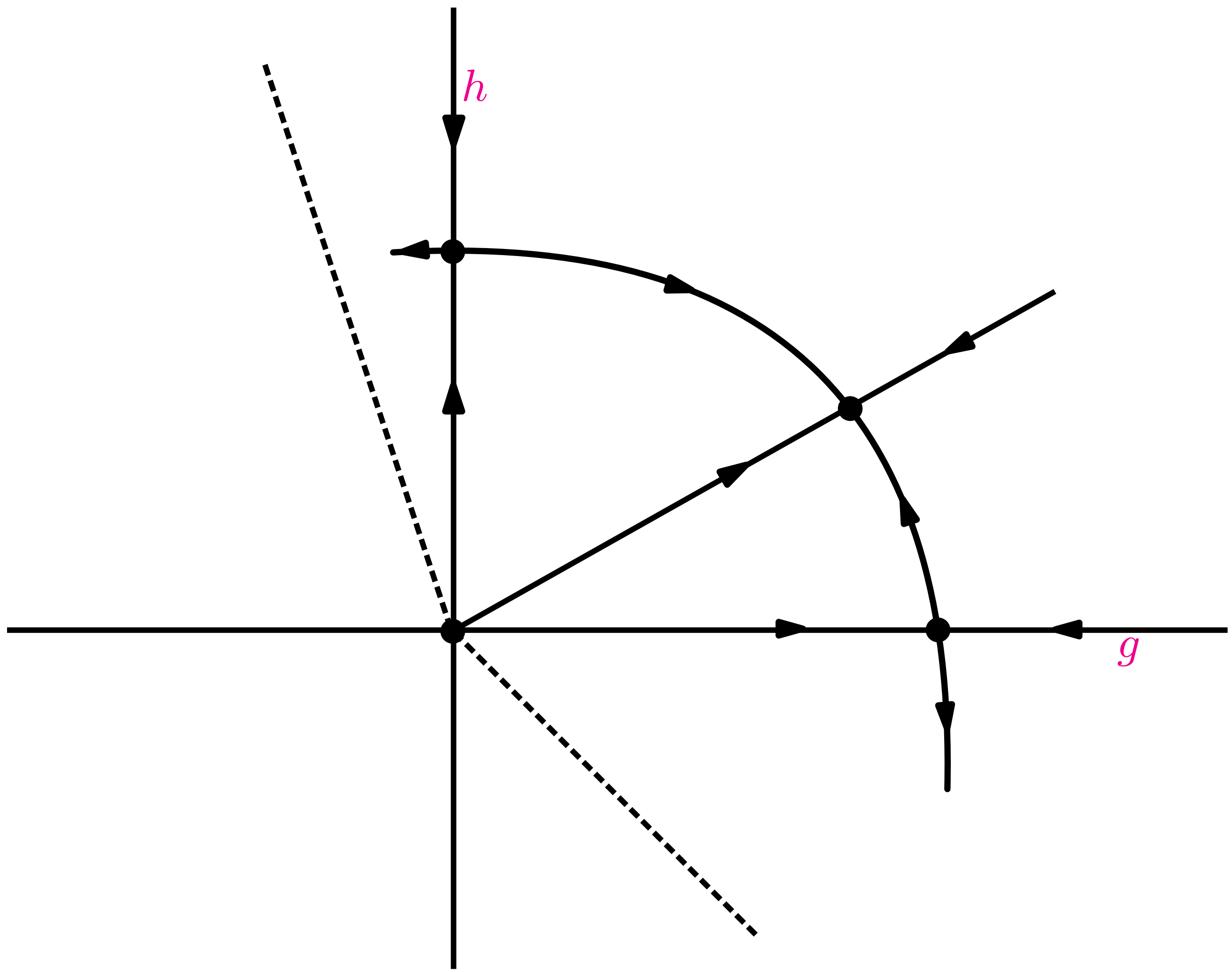

Fig. 8 Cubic anisotropy: RG flow for $N > 4$. The dotted lines correspond to the positivity conditions.

The global RG flow

Since the RG equations define a unique direction at each point, the RG trajectories can intersect only at fixed points where the direction is undefined.

The lines $h = 0$ and $g = 0$ and the half-line joining the origin to the cubic fixed point are stable lines for the RG, a special case of a general property, and thus cannot be crossed. In the planar case, these conditions fix completely the topology of the RG trajectories (Fig. 8).

In particular, there exist initial parameters g, h, satisfying the positivity conditions, such that the RG trajectories can reach no fixed point but, instead, evolve toward non-positive regions (Fig. 8) where the Hamiltonian is bound by higher powers of σ.

These parameters, generically, correspond to weak first-order transitions: the correlation length remains finite when $T \to T_{c+}$ and, in microscopic units, takes a value that is of the order of the maximum dilatation parameter such that the parameters g, h are still in the allowed region.

Renormalization group: General results

We consider now general field theories that involve an N-component vector field $\boldsymbol{\sigma}(x)$ with quartic interactions, satisfy the condition (55) and thus admit a unique correlation length. We first discuss quite generally the RG flow of the amplitudes of the quartic terms at T_{c} in dimension $d = 4 - \varepsilon$ and derive a few interesting properties.

Hamiltonian flow

We assume that the Hamiltonian contains $p > 1$, linearly independent, quartic terms with coefficients g^a, $a = 1, \ldots, p$. The flow equations that replace the unique flow equation of the $O(N)$ case, then take the general form

$$\lambda \frac{\mathrm{d}g^a}{\mathrm{d}\lambda} = -\beta^a\bigl(g(\lambda)\bigr). \tag{57}$$

With this convention, a dilatation $\lambda \to \infty$ corresponds to the large-distance behaviour.

The flow equation implies that the vector tangent to an RG trajectory at a point g^a is proportional to $\beta^a(g)$. The tangent vector is thus unique in each point where it is defined. This implies that RG trajectories can intersect only at fixed points, solutions of $\beta^a(g) = 0$.

In the framework of the ε-expansion with $g^a = O(\varepsilon)$, at leading order the β-functions can be written as

$$\beta^a(g) = -\varepsilon g^a + B^a(g), \tag{58}$$

where $B^a(g)$ is a homogeneous, second-degree polynomial. It thus satisfies the two identities

$$B^a(\rho g) = \rho^2 B^a(g), \tag{59a}$$

$$\Rightarrow \quad \left.\frac{\mathrm{d}B^a(\rho g)}{\mathrm{d}\rho}\right|_{\rho=1} = \sum_{b=1}^{p} g^b \frac{\partial B^a(g)}{\partial g^b} = 2B^a(g). \tag{59b}$$

Fixed points and stability. In the case of p parameters g^a, the maximum number of real solutions g_*^a of the fixed-point equations

$$\beta^a(g_*) \equiv -\varepsilon g_*^a + B^a(g_*) = 0\,, \tag{60}$$

is 2^p. The local stability of these fixed points can be studied by linearizing the flow equations:

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}(g^a - g_*^a) = \sum_b L_b^a (g^b - g_*^b)$$

with

$$L_b^a = -\frac{\partial \beta^a(g_*)}{\partial g^b} = \varepsilon \delta_b^a - \frac{\partial B^a(g_*)}{\partial g^b}\,.$$

The local stability properties thus depend on the sign of the eigenvalues (we prove later that they are real) of the matrix $\mathbf{L}$ of partial derivatives. If all eigenvalues of $\mathbf{L}$ are negative, the fixed point is locally stable.

Global stability properties depend on the complete solutions of the flow equation, which determine, in the space of parameters g^a, the domain of attraction of each fixed point.

Eigenvalue $-\varepsilon$. From the homogeneity property (59b) and from the fixed--point equation (60), one infers

$$\sum_b L^a_b g^b_* = \varepsilon g^a_* - 2B^a(g_*) = -\varepsilon g^a_* \,.$$

One concludes that all non-Gaussian fixed points have at least one direction of stability corresponding to the eigenvector g^a_*, with the same eigenvalue $-\varepsilon + O(\varepsilon^2)$ at leading order.

Special RG trajectories. A more general result can be derived. One looks for special solutions of the flow equation of the form

$$g^a(\lambda) = \rho(\lambda) g^a_* \,, \quad g_* \neq 0 \,, \quad \rho \geq 0 \,.$$

Introducing the Ansatz into the flow equation, using the fixed-point equation (60) and the homogeneity property (59*a*), one infers

$$\begin{aligned} g^a_* \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \rho(\lambda) = \varepsilon \rho(\lambda) g^a_* - B^a\big(\rho(\lambda) g_*\big) &= \varepsilon \rho(\lambda) g^a_* - \rho^2(\lambda) B^a(g_*) \\ &= \varepsilon \rho(\lambda) g^a_* - \varepsilon g^a_* \rho^2(\lambda). \end{aligned}$$

The flow equation is thus compatible with the Ansatz and the function $\rho(\lambda)$ is a solution of

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \rho(\lambda) = \varepsilon \rho(\lambda)\big(1 - \rho(\lambda)\big).$$

At leading order in the ε-expansion, the half-lines joining the Gaussian fixed point to other fixed points are RG trajectories and on these trajectories non-Gaussian fixed points are stable.

Gradient flow

It has been verified up to order ε^5 (*i.e.*, all known orders) that the RG β-functions of the general models with quartic interaction can be written as

$$\beta^a(g) = \sum_b T^{ab}(g) \frac{\partial U}{\partial g^b}, \tag{61}$$

where the matrix $\mathbf{T}$ with elements T^{ab} is a symmetric positive matrix, and a regular function of the coefficients g^a. The flow equation then defines a gradient flow.

However, let us point out that the regularity and positivity properties of the matrix $\mathbf{T}$ can only be verified, in the perturbative framework, in the vicinity of the Gaussian fixed point $g_* = 0$.

Change of parametrization

The general form (61) is the only form of a gradient flow consistent with transformation properties under reparametrization (diffeomorphisms) in the space of the coefficients g^a.

Indeed, let us introduce new parameters γ^a and change variables, $g^a = g^a(\gamma)$, in the flow equations. The matrix $\partial\gamma^b/\partial g^a$ must be invertible for the mapping $\gamma^a \mapsto g^a$ to be invertible. Within the framework of the ε-expansion, this implies that the matrix must be invertible at $g = 0$.

The chain rule for partial derivatives leads to

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma^a = \sum_b \frac{\partial\gamma^a}{\partial g^b}\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g^b\,,\quad \frac{\partial U}{\partial g^a} = \sum_b \frac{\partial\gamma^b}{\partial g^a}\frac{\partial U}{\partial\gamma^b}\,.$$

Then,

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma^a = -\sum_b T'^{ab}\frac{\partial U}{\partial\gamma^b} \text{ with } T'^{ab} = \sum_{c,d}\frac{\partial\gamma^a}{\partial g^c}T^{cd}\frac{\partial\gamma^b}{\partial g^d}. \qquad (62)$$

One verifies that if the matrix $\mathbf{T}$ is symmetric and positive, the transformed matrix $\mathbf{T}'$ with elements T'^{ab} is also symmetric and positive.

One notes that even if the matrix $\mathbf{T}$ is proportional to the identity matrix in one special parametrization, this is in general no longer true in a different parametrization.

Finally, since the matrix $\mathbf{T}$ is positive and transforms under reparametrization as shown in (62), the elements $[\mathbf{T}^{-1}]_{ab}$ of its inverse have the properties of a metric tensor.

Gradient flow and potential

The property of gradient flow has several consequences:

(i) The potential decreases along an RG trajectory and thus fixed points are extrema of the potential, stable fixed points being local minima.

(ii) All eigenvalues of the matrix of first-order partial derivatives $L^a_b = -\partial\beta^a/\partial g^b$ at a fixed point are real.

Derivation. The variation of the potential U along a trajectory satisfies

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} U\big(g(\lambda)\big) = \sum_a \frac{\partial U}{\partial g^a} \lambda \frac{\mathrm{d}g^a}{\mathrm{d}\lambda} = -\sum_{a,b} \frac{\partial U}{\partial g^a} T^{ab} \frac{\partial U}{\partial g^b} .$$

Since the matrix $\mathbf{T}$ is positive, the right-hand side, which is the expectation value of a negative matrix, is negative. Thus, the potential decreases along a trajectory. The fixed points are extrema of the function U:

$$\beta^a(g) = 0 \;\Leftrightarrow\; \frac{\partial U(g)}{\partial g^a} = 0 .$$

A stable fixed point is a local minimum of $U(g)$.

At a fixed point, the elements of the matrix $\mathbf{L}$ of the derivatives of the RG functions $-\beta$ are given by

$$L^a_b = -\frac{\partial \beta^a(g)}{\partial g^b} = -\sum_c T^{ac}(g) \frac{\partial^2 U(g)}{\partial g^c \partial g^b} \quad \text{or} \quad \mathbf{L} = -\mathbf{T}\mathbf{U}'' . \qquad (63)$$

Since the matrix $\mathbf{T}$ is positive, it can be written as the square of a matrix $\mathbf{X}$, also symmetric and positive:

$$\mathbf{T} = \mathbf{X}^2 , \quad \mathbf{X} > 0 .$$

The matrix

$$\mathbf{M} = \mathbf{X}^{-1}\mathbf{L}\mathbf{X} = -\mathbf{X}\mathbf{U}''\mathbf{X} ,$$

has the same eigenvalues as $\mathbf{L}$, but since the matrices $\mathbf{U}''$ and $\mathbf{X}$ are symmetric, it is a symmetric matrix. The matrix $\mathbf{L}$, which has the same eigenvalues as a real symmetric matrix, thus has also real eigenvalues.

The relation also shows that if the matrix $\mathbf{U}''$ is positive, the matrix $\mathbf{X}\mathbf{U}''\mathbf{X}$ is positive (and conversely), and the corresponding fixed point thus is locally stable.

Fixed points, gradient flow and stability

In the framework of the ε-expansion, we now prove two other consequences of the property of gradient flow: there exists at most one stable fixed point; the stable fixed point corresponds to the lowest value of the potential.

Indeed, let us assume the existence of two fixed points corresponding to the parameters g_1 and g_2. We then consider the parameters g of the form

$$g^a(s) = s g_1^a + (1-s) g_2^a\,, \quad 0 \le s \le 1\,,$$

and the corresponding potential $u(s) = U\big(g(s)\big)$.

Note that the coefficients g^a must satisfy some positivity condition for the Hamiltonian to be positive for $|\boldsymbol{\sigma}| \to \infty$ (see condition (64)) giving to the g^a space a structure of convex cone. This condition, which can be shown to be satisfied by all fixed points and thus for $s=0$ and $s=1$, is then also verified for all parameters $g^a(s)$ such that $0 \le s \le 1$.

Since at leading order the β-functions are quadratic, $u(s)$ is a third-degree polynomial, as the explicit form (72) will confirm. The derivative

$$u'(s) = \sum_a g'^a(s) \frac{\partial U}{\partial g^a} = \sum_a (g_1^a - g_2^a) \frac{\partial U}{\partial g^a} = \sum_{a,b} (g_1^a - g_2^a) T_{ab}^{-1} \beta^b \big(g(s)\big)$$

vanishes due to the fixed-point conditions at $s = 0$ and $s = 1$:

$$u'(0) = u'(1) = 0 \,.$$

Since $u'(s)$ is a second-degree polynomial, it then necessarily has the form

$$u'(s) = As(1-s).$$

The second derivative $u''(s)$ is given in terms of the matrix of second partial derivatives of U and, thus, the partial derivatives of the RG functions β, by

$$u''(s) = \sum_{a,b} (g_1^a - g_2^a) \frac{\partial^2 U\big(g(s)\big)}{\partial g^a \partial g^b} (g_1^b - g_2^b) = A(1-2s) \,.$$

In particular, for $s=0$ and $s=1$,

$$A=\sum_{a,b}(g_1^a-g_2^a)\frac{\partial^2 U(g_2)}{\partial g^a \partial g^b}(g_1^b-g_2^b)\,, \quad -A=\sum_{a,b}(g_1^a-g_2^a)\frac{\partial^2 U(g_1)}{\partial g^a \partial g^b}(g_1^b-g_2^b)\,.$$

But at a stable fixed point the matrix $\mathbf{U}''$ of partial second derivatives of U is positive. Thus, if g_1 and g_2 are stable fixed points, A and $-A$ are both given by the expectation value of a positive matrix and thus are both positive, which is contradictory: the two fixed points cannot both be stable.

More generally, the sign of A characterizes, in some sense, the relative stability of these two fixed points. Let us assume, for example, that $A<0$ which is consistent with the assumption that g_1 is stable. Then $u'(s)<0$ in $[0,1]$ and $U(g(s))$ is a decreasing function. Thus,

$$U(g_1)<U(g_2).$$

In particular, if g_1 is a stable fixed point, it corresponds, among all fixed points, to the lowest value of the potential.

Explicit general expressions: RG analysis

A general Hamiltonian satisfying all assumptions can be written as

$$\mathcal{H}(\boldsymbol{\sigma}) = \int \mathrm{d}^d x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[\sum_{\mu=1}^{d} (\partial_\mu \sigma_i)^2 + (r_\mathrm{c} + \tau)\, \sigma_i^2 \right] + \frac{1}{4!} \sum_{i,j,k,l=1}^{N} g_{ijkl} \sigma_i \sigma_j \sigma_k \sigma_l \right\},$$

where g_{ijkl} is a tensor symmetric in its four indices i, j, k, l. (we have omitted the regularizing terms.)

The Hamiltonian $\mathcal{H}$ must be positive for $|\sigma| \to \infty$ for the phase transition to be continuous. This implies the condition for g_{ijkl},

$$\sum_{i,j,k,l} g_{ijkl} \sigma_i \sigma_j \sigma_k \sigma_l > 0 \quad \forall \sigma_i \text{ such that } |\sigma| = 1\,. \tag{64}$$

The positivity condition gives to the space of admissible parameters g the structure of a convex cone.

Moreover, since the connected two-point correlation function in the disordered phase must be proportional to the identity matrix,

$$\langle\sigma_i(x)\sigma_j(y)\rangle_{\text{conn.}} = W^{(2)}_{ij}(x,y) = \delta_{ij}W^{(2)}(x,y), \tag{65}$$

the tensor g_{ijkl} has special properties that take the form of successive constraints on the tensor g_{ijkl} in the perturbative expansion.

RG functions

The flow equation for the parameters $g_{ijkl}(\lambda)$ in the general Hamiltonian now reads

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}g_{ijkl}(\lambda) = -\beta_{ijkl}\big(g(\lambda)\big). \tag{66}$$

The large-distance behaviour of the field theory is governed by fixed points. These are solutions of the equation

$$\beta_{ijkl}(g^*) = 0\,.$$

The local stability properties of fixed points are related to the eigenvalues of the matrix

$$L^*_{ijkl,i'j'k'l'} = -\frac{\partial \beta_{ijkl}(g^*)}{\partial g_{i'j'k'l'}}. \tag{67}$$

RG functions as ε-expansions. It is simple to calculate the RG functions corresponding to a general σ^4 type theory. As in the example of the $O(N)$ symmetry, calculations differ from the case $N = 1$ only by geometric factors.

The RG β-function, at leading order, is given by

$$\beta_{ijkl}(g) = -\varepsilon g_{ijkl} + \frac{1}{16\pi^2} \sum_{m,n} \left(g_{ijmn} g_{mnkl} + g_{ikmn} g_{mnjl} + g_{ilmn} g_{mnkj} \right) + O\left(g^3\right). \tag{68}$$

The field dimension can be inferred from the function

$$\eta(g) = \frac{1}{6N(4\pi)^4} \sum_{i,j,k,l} g_{ijkl} g_{ijkl} + O\left(g^3\right), \tag{69}$$

a result that satisfies, in particular, the condition $\eta \geq 0$ derived from quantum field theory.

The flow equation for the deviation τ of the critical temperature can be written as

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \tau(\lambda) = \frac{1}{\nu\big(g(\lambda)\big)} \text{ with } \frac{1}{\nu(g)} = 2 - \frac{1}{16\pi^2 N} \sum_{i,j} g_{iijj} + O\left(g^2\right). \tag{70}$$

In the two equations (69) and (70), we have used explicitly the condition (65), which implies

$$\sum_{k} g_{ijkk} = \frac{\delta_{ij}}{N} \sum_{k,l} g_{kkll}, \quad \sum_{k,l,m} g_{iklm} g_{jklm} = \frac{\delta_{ij}}{N} \sum_{k,l,m,n} g_{klmn} g_{klmn}. \tag{71}$$

Moreover, at a fixed point

$$\begin{aligned}\varepsilon \sum_{i,j,k,l} g_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l &= \frac{3}{16\pi^2} \sum_{i,j,k,l,m,n} \sigma_i\sigma_j g_{ijmn} g_{mnkl}\sigma_k\sigma_l \\ &= \frac{3}{16\pi^2} \sum_{m,n} \left(\sum_{i,j} \sigma_i\sigma_j g_{ijmn} \right)^2 .\end{aligned}$$

Thus, any non-Gaussian fixed point satisfies the positivity condition (64).

Stability of the isotropic fixed point

Among the possible fixed points, one always finds, in addition to the Gaussian fixed point, the fixed point corresponding to the $O(N)$ symmetric Hamiltonian. It is possible to study its local stability at leading order in ε.

One can first specialize the expressions (68)–(70) to the example of the $(\boldsymbol{\sigma}^2)^2$ field theory with $O(N)$ symmetry. This amounts to substituting in these equations

$$g_{ijkl} = \frac{g}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

After a short calculation, one recovers the expressions for the $\beta(g)$ and $\eta(g)$ functions of the $O(N)$ model and the corresponding value of g^*.

The stability conditions are given by the eigenvalues of the matrix $\mathbf{L}^*$ (equation (67)). Setting

$$g_{ijkl} = g^*_{ijkl} + s_{ijkl} \,,$$

at leading order one finds

$$(L^* s)_{ijkl} = \varepsilon s_{ijkl} - \frac{\varepsilon}{N+8}\left(\delta_{ij}\sum_m s_{mmkl} + 5 \text{ terms} + 12 s_{ijkl}\right),$$

where the five terms are obtained by permutation of the indices i, j, k, l.

Taking s_{ijkl} proportional to g^*_{ijkl}, one recovers the exponent $\omega = \beta'(g^*)$ of the isotropic model. More generally, the eigenvectors can be classified according to their trace properties.

We thus parametrize s_{ijkl} in the form

$$s_{ijkl} = ug^*_{ijkl} + (v_{ij}\delta_{kl} + 5 \text{ terms}) + w_{ijkl} \,,$$

where the tensors v_{ij} and w_{ijkl} are traceless: $\sum_i v_{ii} = 0$, $\sum_k w_{ijkk} = 0$.

The three eigenvalues corresponding to the components u, w, v are, respectively, $-\omega$, $-\omega_{\text{anis.}}$ and $-\omega'$ with

$$\omega = \varepsilon + O\left(\varepsilon^2\right), \; \omega_{\text{anis.}} = \varepsilon \frac{4-N}{N+8} + O\left(\varepsilon^2\right), \quad \omega' = \frac{8\varepsilon}{N+8} + O\left(\varepsilon^2\right).$$

The perturbation proportional to v_{ij} does not satisfy the first condition (71). It lifts the degeneracy between the correlation lengths of the various field components. This induces a crossover to a situation where several components of the field decouple. However, one easily verifies that the corresponding eigenvalue ω' produces, for ε small, sub-leading effects compared to the eigenvalue corresponding to the quadratic operator $\sigma_i\sigma_j$.

For the set of models satisfying the condition (71), the leading eigenvalue is ω_{anis}. One then finds the following interesting result (which generalizes a result obtained in the example of the cubic anisotropy): the $O(N)$ symmetric fixed point is stable with respect to all perturbations for N smaller than a value N_{c}. The calculation of $\omega_{\text{anis.}}$ at order ε yields

$$N_{\text{c}} = 4 - 2\varepsilon + O\left(\varepsilon^2\right).$$

This is a new example of a symmetry generated dynamically: for $N < N_{\text{c}}$, in the critical domain, correlation functions have a large-distance behaviour that exhibits more symmetry than the initial microscopic theory.

Gradient flow: Fixed points, stability and field dimension

One verifies that the general expression of the β-function derives from a potential:

$$\beta_{ijkl}(g) = \frac{\partial U(g)}{\partial g_{ijkl}}$$

with

$$U(g) = -\frac{\varepsilon}{2} \sum_{i,j,k,l} g_{ijkl} g_{ijkl} + \frac{1}{(4\pi)^2} \sum_{i,j,k,l,m,n} g_{ijkl} g_{klmn} g_{mnij} \,. \tag{72}$$

Fixed-point stability and exponent η. In the framework of the ε-expansion, we now prove the intriguing result: the stable fixed point (or at least the stablest) corresponds to the largest value of the dimension of the field σ and thus to the correlation functions that have the fastest decay at large distance.

For any fixed point g^*, the equations $\beta = 0$ imply the relation

$$\sum_{i,j,k,l} g^*_{ijkl}\beta_{ijkl} = 0 \;\Rightarrow\; \varepsilon \sum_{i,j,k,l} g^*_{ijkl}g^*_{ijkl} = \frac{3}{(4\pi)^2} \sum_{i,j,k,l,m,n} g^*_{ijkl}g^*_{klmn}g^*_{mnij} \,.$$

Using the relation to simplify $U(g^*)$, one obtains the expression

$$U(g^*) = -\tfrac{1}{6}\varepsilon \sum_{i,j,k,l} g^*_{ijkl}g^*_{ijkl} + O(g^4).$$

Moreover, at leading order the exponent η is given by equation (69):

$$\eta = \frac{1}{6N}\frac{1}{(4\pi)^4} \sum_{i,j,k,l} g^*_{ijkl}g^*_{ijkl} = -\frac{1}{N\varepsilon}\frac{1}{(4\pi)^4}U(g^*). \qquad (73)$$

As we have shown, the stable fixed point corresponds to the lowest value of U. It thus corresponds also to the largest value of the exponent η and thus of the dimension of the field σ.

The validity of this result, beyond the ε-expansion, remains a conjecture, though a few positive numerical checks have been reported.

Exercises

Exercise 13

Cubic anisotropy. Using the explicit expressions of the β-functions (56) for the Hamiltonian with cubic anisotropy,

$$\beta_g(g,h) = -\varepsilon g + \frac{1}{8\pi^2}\left(\frac{N+8}{6}g^2 + gh\right),$$

$$\beta_h(g,h) = -\varepsilon h + \frac{1}{8\pi^2}\left(2gh + \frac{3}{2}h^2\right),$$

find the matrices $\mathbf{T}$ such that one can write

$$\frac{\partial U}{\partial g} = T_{11}\beta_g + T_{12}\beta_h$$

$$\frac{\partial U}{\partial h} = T_{21}\beta_g + T_{22}\beta_h\,,$$

and determine the corresponding potential function U. Calculate the values of the potential for the different fixed points and discuss.