RG equations in the critical domain above T_c

Correlation functions also exhibit universal properties near T_c in the critical domain where the correlation length ξ is large in the microscopic scale, here, $\xi\Lambda \gg 1$.

To describe these universal properties above T_c, one adds to the critical Hamiltonian the σ^2 relevant term:

$$\mathcal{H}_\tau(\sigma) = \mathcal{H}_c(\sigma) + \frac{\tau}{2}\int \mathrm{d}^d x\, \sigma^2(x),$$

where $\tau \propto T - T_c \ll \Lambda^2$ characterizes the deviation from the critical temperature.

The extended renormalization theorem involves a new renormalization factor $Z_2(\Lambda/\mu, g)$, ratio between the full renormalization of $\int \mathrm{d}^d x\, \sigma^2(x)$ and the Gaussian renormalization.

One then derives a more general RG equation of the form (Zinn-Justin 1973)

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\frac{n}{2}\eta(g)-\eta_2(g)\tau\frac{\partial}{\partial\tau}\right]\tilde{W}^{(n)}_{\text{as.}}(p_i;\tau,g,\Lambda)=0\,,$$

where a new RG function $\eta_2(g)$, related to $Z_2(\Lambda/\mu,g)$, appears.

These equations can be further generalized to include an external field (a magnetic field for magnetic systems) and the corresponding induced field expectation value (magnetization for magnetic systems). An RG equation for the equation of state follows.

Renormalized RG equations

For $d = 4 - \varepsilon$, if one is interested only in the leading scaling behaviour (and the first correction), it is technically simpler to use dimensional regularization and the renormalized theory in the so-called minimal (or modified minimal) subtraction scheme. The relation between initial and renormalized correlation functions is asymptotically symmetric. One thus derives also (for the critical theory)

$$\left(\mu\frac{\partial}{\partial\mu} + \tilde{\beta}(g_{\rm r})\frac{\partial}{\partial g_{\rm r}} + \frac{n}{2}\tilde{\eta}(g_{\rm r})\right)\tilde{W}_{\rm r}^{(n)}(p_i, g_{\rm r}, \mu) = 0$$

with the definitions

$$\tilde{\beta}(g_{\rm r}) = \mu\frac{\partial}{\partial\mu}\bigg|_g g_{\rm r}\,, \quad \tilde{\eta}(g_{\rm r}) = \mu\frac{\partial}{\partial\mu}\bigg|_g \ln Z(g_{\rm r}, \varepsilon)\,.$$

In this scheme, the renormalization constants are obtained by continuation to low dimensions where the infinite Λ limit, at $g_{\rm r}$ fixed, can be taken.

For example,

$$\lim_{\Lambda\to\infty} Z(\Lambda/\mu, g)\big|_{g_{\mathrm{r}}\ \text{fixed}} = Z(g_{\mathrm{r}}, \varepsilon).$$

Then, order by order in powers of g_{r}, they have a Laurent expansion in powers of ε. In the minimal subtraction scheme, the freedom in the choice of renormalization constants is used to reduce the Laurent expansion to the singular terms. For example, $Z(g_{\mathrm{r}}, \varepsilon)$ takes the form

$$Z(g_{\mathrm{r}}, \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{\sigma_n(g_{\mathrm{r}})}{\varepsilon^n} \text{ with } \sigma_n(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^{n+1}).$$

A remarkable consequence is that the RG functions $\tilde{\eta}(g_{\mathrm{r}})$, and $\tilde{\eta}_2(g_{\mathrm{r}})$ when a σ^2 term is added, become independent of ε and $\tilde{\beta}(g_{\mathrm{r}})$ has the simple dependence $\tilde{\beta}(g_{\mathrm{r}}) = -\varepsilon g_{\mathrm{r}} + \tilde{\beta}_2(g_{\mathrm{r}})$, where $\tilde{\beta}_2(g_{\mathrm{r}}) = O(g_{\mathrm{r}}^2)$ is also independent of ε.

Remark. Strong arguments indicate that the renormalized theory exists only for $0 \le g_{\mathrm{r}} \le g_{\mathrm{r}}^*$, where g_{r}^* is the IR fixed point: $\tilde{\beta}(g_{\mathrm{r}}^*) = 0$, $\tilde{\beta}'(g_{\mathrm{r}}^*) > 0$.

Solution of the RG equations: The epsilon-expansion

RG equations can be solved by the method of characteristics.

In the simplest example of the critical theory, one introduces a scale parameter λ and two functions $g(\lambda)$ and $\zeta(\lambda)$ defined by

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda) = -\beta\big(g(\lambda)\big),\ g(1) = g\,, \quad \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \zeta(\lambda) = -\eta\big(g(\lambda)\big),\ \zeta(1) = 1\,.$$

The function $g(\lambda)$ is the effective coefficient of the σ^4 term at the scale λ. The differential RG equation then implies

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \left[\zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda/\lambda\big)\right] = 0\,,$$

from which one infers

$$\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g, \Lambda\big) = \zeta^{n/2}(\lambda) \tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda/\lambda\big).$$

Rescaling $\Lambda \mapsto \lambda\Lambda$, the equation becomes

$$\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g, \lambda\Lambda\big) = \zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda\big).$$

From its definition, one verifies that $\tilde{W}^{(n)}_{\text{as.}}$ has dimension $(d - (d+2)n/2)$. Therefore,

$$\begin{aligned}\tilde{W}^{(n)}_{\text{as.}}\big(p_i/\lambda; g, \Lambda\big) &= \lambda^{(d+2)n/2-d}\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g, \lambda\Lambda\big)\\ &= \lambda^{(d+2)n/2-d}\zeta^{n/2}(\lambda)\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g(\lambda), \Lambda\big).\end{aligned}$$

These equations show that the general Hamiltonian flow reduces here to the flow of $g(\lambda)$ and, thus, the zeros of the function $\beta(g)$ determine the fixed points that govern the large distance behaviour.

Since

$$\beta(g) = -\varepsilon g + O(g^2),$$

when $\lambda \to \infty$, if $g > 0$ is initially very small, $g(\lambda)$ moves away from the unstable Gaussian fixed point, in agreement with the general RG analysis of the Gaussian fixed point.

If one assumes the existence of another, attractive zero $g^* > 0$ with then $\beta'(g^*) > 0$, $g(\lambda)$ will converge toward g^*. If $\eta(g^*) \equiv \eta$ is finite, one finds the universal behaviour

$$\tilde{W}^{(n)}_{\text{as.}}\big(p_i/\lambda; g, \Lambda\big) \underset{\lambda\to\infty}{\propto} \lambda^{(d+2-\eta)n/2-d}\tilde{W}^{(n)}_{\text{as.}}\big(p_i; g^*, \Lambda\big).$$

For the connected correlation functions in position space, this result translates into

$$W^{(n)}_{\text{as.}}\big(\lambda x_i; g, \Lambda\big) \underset{\lambda\to\infty}{\propto} \lambda^{-n(d-2+\eta)/2}W^{(n)}_{\text{as.}}\big(x_i; g^*, \Lambda\big),$$

for all x_i distinct.

The exponent $d_\sigma = (d-2+\eta)/2$ is the scaling dimension of the field σ, from the point of view of large distance properties at the critical temperature.

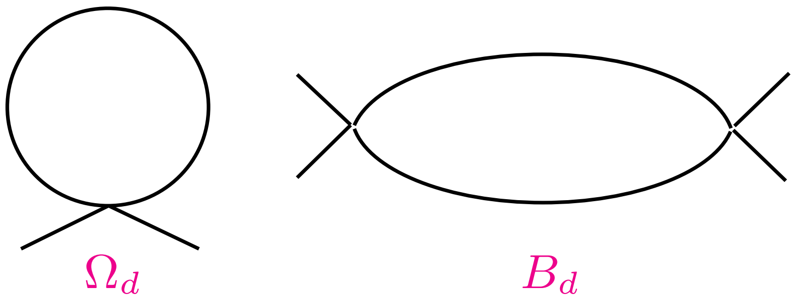

Fig. 6 σ^4 field theory: two one-loop diagrams.

The RG functions at one-loop

We now determine the RG functions at one-loop order by calculating the two and four vertex functions at one-loop order and inserting them in the RG equations.

The inverse or vertex two-point function. At one-loop order,

$$\tilde{\Gamma}^{(2)}(p) = p^2 + r + \tfrac{1}{2} g\Omega_d + O(g^2),$$

where Ω_d is a constant (first diagram of Fig. 6). The theory is critical if $\tilde{\Gamma}^{(2)}(0) = 0$, a condition that determines the critical value of the parameter r at order g: $r = r_c(g) \equiv -\frac{1}{2} g\Omega_d + O(g^2)$.

Then, $\tilde{\Gamma}^{(2)}(p) = p^2 + O(g^2)$. Since β is of order g and $\tilde{\Gamma}^{(2)}$ satisfies the RG equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta(g)\right)\tilde{\Gamma}^{(2)}(p;g,\Lambda) = 0\,,$$

one concludes $\eta(g) = O(g^2)$

The four-point vertex (or 1PI) function. At one-loop order,

$$\tilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) = \Lambda^{\varepsilon}g - \tfrac{1}{2}g^2\Lambda^{2\varepsilon}\left[B_d(p_1+p_2) + B_d(p_1+p_3)\right.$$
$$\left. + B_d(p_1+p_4)\right] + O(g^3),$$

where B_d is the second diagram of Fig. 6:

$$B_d(p) = \frac{1}{(2\pi)^d}\int \mathrm{d}^d q\, \tilde{\Delta}(q)\tilde{\Delta}(p-q) \underset{\Lambda\to\infty}{\sim} \frac{1}{(2\pi)^d}\int_{1<|q|<\Lambda}\frac{\mathrm{d}^d q}{q^4}$$
$$\underset{\Lambda\to\infty}{\sim} \frac{1}{8\pi^2}\left[\ln\Lambda + O(1)\right] + O(\varepsilon).$$

Thus, $\tilde{\Gamma}^{(4)} = g + g\varepsilon \ln \Lambda - \dfrac{3g^2}{16\pi^2} \ln \Lambda + O(g^2) \times 1 + O(g^3, g^2\varepsilon)$.

The four-point vertex function satisfies

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - 2\eta(g)\right) \tilde{\Gamma}^{(4)}(p_i; g, \Lambda) = 0\,.$$

Inserting the perturbative expression, using $\eta(g) = O(g^2)$, one obtains

$$\beta(g, \varepsilon) = -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3, g^2\varepsilon).$$

The RG β-function and the IR fixed point. Using the perturbative calculation of the two- and four-point functions at one-loop order, one has thus derived

$$\beta(g,\varepsilon) = -\varepsilon g + \frac{3}{16\pi^2} g^2 + O(g^3, \varepsilon g^2).$$

In the sense of an ε-expansion, $\beta(g)$ has a zero g^* with a positive slope (Wilson–Fisher 1972),

$$g^* = \frac{16\pi^2 \varepsilon}{3} + O(\varepsilon^2), \quad \omega = \beta'(g^*) = \varepsilon + O(\varepsilon^2),$$

which is an attractive fixed point and thus governs the low momentum behaviour of correlation functions. Moreover, the exponent ω governs the leading correction to the critical behaviour.

Inserting the expansion of $g^*(\varepsilon)$ in the perturbative expansions of other RG functions, one determines the ε-expansion of other critical exponents or universal functions.

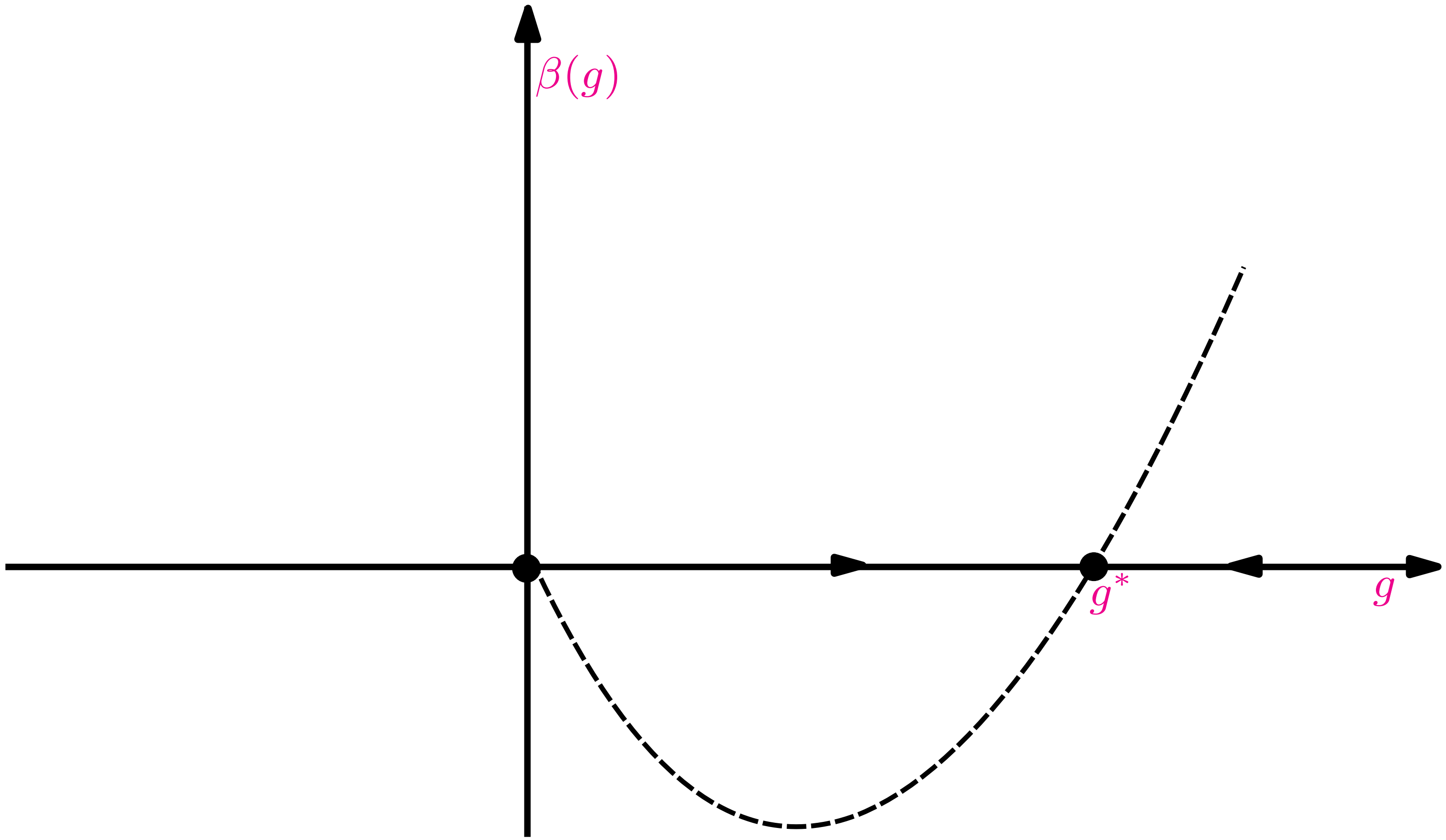

Fig. 7 The RG β-function and RG flow in the $(\boldsymbol{\sigma}^2)^2$ field theory for $d < 4$.

The critical domain. Calculating with a small deviation from criticality, $r = r_c + \tau$, one finds

$$\tilde{\Gamma}^{(2)}(p=0) = \tau + \frac{g\Lambda^{\varepsilon}}{2(2\pi)^d} \int \mathrm{d}^d k \left[\tilde{\Delta}(k,\tau) - \tilde{\Delta}(k,0)\right] + O(g^2).$$

Thus, for $\Lambda \to \infty$,

$$\begin{aligned} \frac{\partial}{\partial\tau}\tilde{\Gamma}^{(2)}(p=0) &= 1 - \frac{g\Lambda^{\varepsilon}}{2(2\pi)^d} \int \mathrm{d}^d k\, \tilde{\Delta}^2(k,\tau) + O(g^2) \\ &= 1 - \frac{g}{16\pi^2}[\ln(\Lambda/\sqrt{\tau}) + O(1)] + O(g^2, g\varepsilon). \end{aligned}$$

Using an RG equation in the critical domain, which at leading order reduces to

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \eta_2(g)\tau\frac{\partial}{\partial\tau} - \eta_2(g)\right]\frac{\partial}{\partial\tau}\tilde{\Gamma}^{(2)}(p=0) = O(g^2, g\varepsilon)\,,$$

one concludes

$$\eta_2(g) = -\frac{g}{16\pi^2} + O(g^2).$$

Generalization: $O(N)$ symmetric models

The results obtained for models with a $\mathbb{Z}_2$ reflection symmetry can easily be generalized to N-vector models with $O(N)$ orthogonal symmetry but, in contrast with the Gaussian model prediction, different values of N correspond to different universality classes.

Their universal properties can then be derived from a field theory with an N-component field $\boldsymbol{\sigma}(x)$ and an $O(N)$ symmetric Hamiltonian of the form

$$\mathcal{H}(\boldsymbol{\sigma}) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \boldsymbol{\sigma}(x)\right)^2 + \tfrac{1}{2} r \boldsymbol{\sigma}^2(x) + \frac{g}{4!}\left(\boldsymbol{\sigma}^2(x)\right)^2\right] + \text{higher derivatives.}$$

The RG β-function becomes

$$\beta(g) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O(g^3, \varepsilon g^2) \ \Rightarrow \ g^* = \frac{48\pi^2}{N+8}\varepsilon + O(\varepsilon^2).$$

Moreover, one finds

$$\eta(g) = O(g^2), \ \eta_2(g) = -\frac{(N+2)g}{48\pi^2} + O(g^2).$$

Finally, the $O(N)$ vector model can be solved exactly in the large N limit and the general predictions of the ε-expansion can then be verified in this limit for all dimensions ($1/N$ corrections can also be evaluated).

Further generalizations involve theories with N-component fields but with symmetry groups subgroup of $O(N)$, such that only one quadratic invariant but several independent quartic σ^4 terms are allowed. The structure of fixed points may then be more complicate as the discussion in a later section illustrates.

Epsilon-expansion: A few general results

First, from the mere existence of a fixed point and of the corresponding ε-expansion, universal properties of an important class of critical phenomena can be proved to all orders in ε: this includes relations between critical exponents (only two are independent), scaling behaviour of correlation functions or of the equation of state.

The scaling equation of state

The scaling properties of the equation of state of magnetic systems, relation between applied magnetic field H, magnetization M and temperature T, provide an example of the general results that can be obtained. In the relevant limit $|H| \ll 1$, $|T - T_c| \ll 1$, using RG arguments, one proves Widom's conjectured scaling form

$$H = M^{\delta} f\big((T - T_c)/M^{1/\beta}\big),$$

where $f(z)$ is a universal function (up to z and f normalizations).

Moreover, the exponents satisfy the relations ($\eta = \eta(g^*)$)

$$\delta = \frac{d+2-\eta}{d-2+\eta}, \quad \beta = \tfrac{1}{2}\nu(d-2+\eta) = \nu d_\sigma \,,$$

where ν, the correlation length exponent, given by $\nu = 1/\big(\eta_2(g^*) + 2\big)$, characterizes the divergence ξ of the correlation length at T_c:

$$\xi \propto |T - T_c|^{-\nu}.$$

Other relations can be derived, involving the magnetic susceptibility exponent γ characterizing the divergence of the two-point correlation function at zero momentum at T_c, or the exponent α characterizing the behaviour of the specific heat:

$$\gamma = \nu(2-\eta), \quad \alpha = 2 - \nu d \,.$$

Note that the relations involving the dimension d explicitly are not valid for the Gaussian fixed point.

Critical exponents as ε-expansions

Universal quantities can be calculated as ε-expansions. As an illustration, we give here two terms of the expansion of the exponents η, γ and ω for the $O(N)$ symmetric $(\boldsymbol{\sigma}^2)^2$ theory, although the RG functions of the field theory are known to five-loop order and, thus, critical exponents are known up to order ε^5.

In terms of the variable $v = N_d g$ where $N_d = 2/(4\pi)^{d/2}\Gamma(d/2)$ is the usual loop factor, the RG functions $\beta(v)$ and $\eta_2(v)$ at two-loop order, $\eta(v)$ at three-loop order, are

$$\beta(v) = -\varepsilon v + \frac{(N+8)}{6}v^2 - \frac{(3N+14)}{12}v^3 + O(v^4),$$

$$\eta(v) = \frac{(N+2)}{72}v^2\left(1 - \frac{(N+8)}{24}v\right) + O(v^4),$$

$$\eta_2(v) = -\frac{(N+2)}{6}v\left(1 - \frac{5}{12}v\right) + O(v^3).$$

The fixed point value solution of $\beta(v^*) = 0$ is then

$$v^*(\varepsilon) = \frac{6\varepsilon}{(N+8)}\left[1 + \frac{3(3N+14)}{(N+8)^2}\varepsilon\right] + O(\varepsilon^3).$$

The values of the critical exponents

$$\eta = \eta(v^*), \quad \gamma = \frac{2-\eta}{2+\eta_2(v^*)}, \quad \omega = \beta'(v^*),$$

follow

$$\eta = \frac{\varepsilon^2(N+2)}{2(N+8)^2}\left[1 + \frac{(-N^2+56N+272)}{4(N+8)^2}\varepsilon\right] + O(\varepsilon^4),$$
$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\varepsilon + \frac{(N+2)}{4(N+8)^3}\left(N^2+22N+52\right)\varepsilon^2 + O(\varepsilon^3),$$
$$\omega = \varepsilon - \frac{3(3N+14)}{(N+8)^2}\varepsilon^2 + O(\varepsilon^3).$$

Though this may not be obvious on these few terms, the ε-expansion is divergent for any $\varepsilon > 0$, as estimates of the large order behaviour of perturbation series based on instanton calculus have demonstrated. For example, adding simply the known successive terms for $\varepsilon = 1$ and $N = 1$ yields

$$\gamma = 1.000\ldots\,,\ 1.1666\ldots\,,\ 1.2438\ldots\,, 1.1948\ldots\,,\ 1.3384\ldots\,,\ 0.8918\ldots\,,$$

the best estimate being provided by summing only up to order ε^2 since the best field theory estimate is $\gamma = 1.2396 \pm 0.0013$.

Divergent series do not define a unique analytic function in general. Extracting more precise estimates from the known terms of the series thus requires an assumption concerning its Borel summability and a practical summation method.

Numerical estimates of exponents from a summation of the ε-expansion

We display below (Table 1) the results for the critical exponents γ, ν, η, β and the correction exponent ω of the $O(N)$ model obtained from a Borel summation of the ε-expansion (Guida and Zinn-Justin 1998), assuming Borel summability. Due to scaling relations like $\gamma = \nu(2 - \eta)$, $\gamma + 2\beta = \nu d$, only two among the first four are independent, but the series have been summed independently to check consistency and precision.

We recall that $N = 0$ corresponds to statistical properties of polymers (mathematically the self-avoiding random walk), $N = 1$ to the Ising universality class, which includes liquid-vapour, binary mixtures or anisotropic magnet phase transitions. $N = 2$ describes the superfluid Helium transition, while $N = 3$ corresponds to isotropic ferromagnets.

Table 1

Critical exponents of the $O(N)$ model for $d = 3$, obtained from the ε-expansion.

N	0	1	2	3
γ	1.1571 ± 0.0030	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
ν	0.5878 ± 0.0011	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
η	0.0315 ± 0.0035	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
β	0.3032 ± 0.0014	0.3265 ± 0.0015	0.3465 ± 0.0035	0.3655 ± 0.0035
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018

Table 2

Critical exponents of the $O(N)$ model for $d = 3$, obtained from fixed $d = 3$ series.

N	0	1	2	3
g_r^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013

As a comparison, we display (Table 2) the best available field theory results, obtained by Borel summation of $d = 3$ renormalized perturbative series (Le Guillou and Zinn-Justin 1980, Guida and Zinn-Justin 1998), based on the Callan–Symanzik formalism, following an initial suggestion of Parisi.

The Callan–Symanzik (CS) formalism. In the CS formalism, the renormalized vertex functions are defined by the conditions

$$\tilde{\Gamma}_{\mathrm{r}}^{(2)}(p) = m^2 + p^2 + O(p^4),$$
$$\tilde{\Gamma}_{\mathrm{r}}^{(4)}(0,0,0,0) = m^{4-d} g_{\mathrm{r}} \,.$$

The limit of interest corresponds to a divergent correlation length $1/m$ and thus $|p \gg m$. One could think that this limit is governed by an UV fixed point, here the Gaussian fixed point.

However, the relation between effective coupling constant g_{r} at physical scale m and the initial coupling constant g at cut-off scale Λ takes the form

$$g\Lambda^{4-d} = G_d(g_r) m^{4-d},$$

where the function G_d has a finite $\Lambda \to \infty$ limit for $d < 4$, except at an IR fixed point. Therefore, the condition $\Lambda \gg m$ fixes g_{r} at the IR stable zero g_{r}^* of the β-function of the renormalized theory. Beyond the ε-expansion, at d fixed, this zero has to be determined numerically.

Exercise 9

Critical phenomena in the large N limit. One considers a model involving an N-component field $\boldsymbol{\phi}$ and an $O(N)$ symmetric Hamiltonian of the form

$$\mathcal{S}(\boldsymbol{\phi}, \lambda) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[(\nabla_x \boldsymbol{\phi}(x))^2 + \lambda(x) \boldsymbol{\phi}^2(x) \right] - \frac{3N}{2g} \left(\lambda(x) - r \right)^2 \right\}, \quad (54)$$

where $\lambda(x)$ is an auxiliary scalar field and the integration runs along the imaginary axis, r and $g > 0$ are parameters.

Eventually, the action has to be regularized by introducing a momentum cut-off.

λ-*integration.* Eliminate the auxiliary field λ, by performing explicitly the Gaussian integration over the field λ, and determine the corresponding $\boldsymbol{\phi}$-field action.

Exercise 10

ϕ-integration Alternatively, integrate over the N-component field ϕ and show that the corresponding effective action has the form (using the general identity $\ln\det = \operatorname{tr}\ln$)

$$\mathcal{S}(\lambda) = N\left\{\tfrac{1}{2}\operatorname{tr}\ln\left[-\nabla_x^2 + \lambda(x)\right] - \frac{3}{2g}\int \mathrm{d}^d x\,(\lambda(x) - r)^2\right\}.$$

Exercise 11

The steepest descent method for large N. In the latter form, it is clear that, for $N \to \infty$, the partition function can be calculated by the steepest descent method. It can be justified that the saddle value of the field $\lambda(x)$ is a constant $\bar{\lambda} = \langle\lambda\rangle = m^2$, where m from the action (54) is the ϕ-field mass. To determine $\bar{\lambda}$, one only needs the action density for constant field.

Justify the expression

$$\frac{\mathcal{S}(\bar{\lambda})}{N \times \text{volume}} = \frac{1}{2(2\pi)^d}\int \mathrm{d}^d p\,\ln(p^2 + \bar{\lambda}) - \frac{3}{2g}(\bar{\lambda} - r)^2.$$

Differentiate with respect to $\bar{\lambda}$ to obtain the gap equation, which determines the $\boldsymbol{\phi}$-field mass or, correspondingly, the correlation length $\xi = 1/m$. Note that the momentum integral has to be regularized by replacing p^2 by a polynomial $\Lambda^2 K(p^2/\Lambda^2) = p^2 + O(p^4)$.

Discuss the solution as a function of the parameter r and the space dimension d. First, identify the transition temperature r_c. Verify that the solutions of the saddle point equations describe only the domain $T \geq T_c$.

Exercise 12

Partial integration. Repeat the exercise by integrating only over $(N-1)$ components of the field $\boldsymbol{\phi}$. First verify that indeed above T_c, $\bar{\lambda} = m^2$ where m is the physical mass of the N field components.

Below T_c, the remaining field component has a non-zero expectation value. Derive the saddle point equations obtained by varying both $\bar{\lambda}$ and the expectation value of $\boldsymbol{\phi}$. Calculate the different $\boldsymbol{\phi}$ two-point functions. Discuss.