The Gaussian fixed point and the linearized RG flow

A RG can be constructed that has the Gaussian model as a fixed point. The Hamiltonian flow can be implemented by the simple scaling

$$\sigma(x) \mapsto \lambda^{(2-d)/2}\sigma(x/\lambda). \tag{49}$$

After the change of variables $x' = x/\lambda$, one verifies that the Hamiltonian

$$\mathcal{H}_{\mathrm{G}}^{*}(\sigma) = \frac{1}{2}\int \mathrm{d}^d x \big(\nabla_x \sigma(x)\big)^2, \tag{50}$$

corresponding to the critical Gaussian model, is invariant. The RG has $\mathcal{H}_{\mathrm{G}}^{*}$ as a fixed point. The Hamiltonian flow (49) describes in fact the general RG in the linear approximation near the Gaussian fixed point.

The linearized RG flow

The transformation (49) generates the linearized RG flow at the Gaussian fixed point. $\mathbb{Z}_2$ symmetric local eigenvectors of the flow are monomials in σ of the form

$$\mathcal{O}_{n,k}(\sigma) = \int \mathrm{d}^d x \, O_{n,k}(\sigma, x) \text{ with } \mathcal{O}_{n,k}(\theta\sigma) = \theta^{2n}\mathcal{O}_{n,k}(\sigma),$$

$O_{n,k}(\sigma, x)$ being a product of powers of the field and its derivatives at point x with $2k$ powers of $\partial_\mu \equiv \partial/\partial x_\mu$.

Dimensions of operators. One defines the dimension of x as -1 of ∂_μ thus as $+1$ and the (Gaussian) dimension of the field σ as $[\sigma] = (d-2)/2$. The dimension $[\mathcal{O}_{n,k}]$ of $\mathcal{O}_{n,k}$ follows:

$$[\mathcal{O}_{n,k}] = -d + n(d-2) + 2k \,. \tag{51}$$

Its RG behaviour under the transformation (49) is then given by a simple dimensional analysis. It can be verified that $\mathcal{O}_{n,k}$ scales like $\lambda^{-[\mathcal{O}_{n,k}]}$, and the corresponding eigenvalue of L^* thus is $\ell_{n,k} = -[\mathcal{O}_{n,k}]$.

Eigenperturbations and eigenvalues

$\ell_{1,0} = 2$: the corresponding eigenvector $\int \mathrm{d}^d x\, \sigma^2(x)$ is relevant: it induces a deviation from the critical temperature and thus a finite correlation length.

$\ell_{1,1} = 0$: the corresponding perturbation $\int \mathrm{d}^d x\, [\nabla_x \sigma(x)]^2$ is a redundant eigenvector since it induces only a finite change of the normalization of $\sigma(x)$.

$\ell_{2,0} = 4 - d$. For $d > 4$, the corresponding eigenvector $\int \mathrm{d}^d x\, \sigma^4(x)$ is irrelevant and no other perturbation is relevant on the critical surface ($\xi = \infty$): the Gaussian fixed point is stable on the critical surface.

At $d = 4$, the eigenvector becomes marginal and below dimension 4 it becomes relevant. In dimension $d = 4 - \varepsilon$, $\varepsilon > 0$ small (a notion we define later), it is the only relevant eigenvector and we will study the RG properties of a Gaussian theory to which this unique term is added.

$\ell_{3,0} = 6 - 2d$: the corresponding operator $\int \mathrm{d}^d x\, \sigma^6(x)$ becomes marginal in $d = 3$ dimension.

Above dimension 2, all other eigenoperators are irrelevant.

To summarize, for systems with a $\mathbb{Z}_2$ or, more generally, with an $O(N)$ symmetry, one concludes that, on the critical surface,

(i) the Gaussian fixed point is stable above space dimension 4;

(ii) by expanding beyond the linearized local flow, one shows that it is marginally stable in dimension 4;

(iii) it is unstable below dimension 4.

Finally, in dimension 3, if by adjusting some parameter the most relevant operator $\int \mathrm{d}^d x\, \sigma^4(x)$ can be suppressed, the flow is governed by the marginal operator $\int \mathrm{d}^d x\, \sigma^6(x)$ and this then corresponds to a tricritical behaviour.

Rescaling and Gaussian renormalization

We now assume that the initial Hamiltonian is very close to the Hamiltonian of the Gaussian fixed point.

The RG flow is then initially very close to the local linear flow. Therefore, we first perform the corresponding RG transformation. We introduce a parameter $\Lambda \gg 1$ and substitute

$$\sigma(x) \mapsto \Lambda^{(2-d)/2} \sigma(x/\Lambda).$$

In QFT, this could be called a Gaussian renormalization.

After the change of variables $x' = x/\Lambda$, a monomial $\mathcal{O}_{n,k}(\sigma)$ is multiplied by $\Lambda^{-[\mathcal{O}_{n,k}]}$, where $-[\mathcal{O}_{n,k}]$ is the associated Gaussian eigenvalue.

After this change, coordinates x have dimension Λ^{-1}, derivatives and momenta dimension Λ and the field dimension $\Lambda^{(d-2)/2}$.

The Hamiltonian as well as by definition the initial parameters of the Hamiltonian are dimensionless.

The Gaussian renormalization can then be inferred from dimensional analysis in terms of Λ. Operators are multiplied by powers of Λ and are relevant, marginal or irrelevant depending whether the coefficient increases, is unity or decreases for $\Lambda \to \infty$.

Cut-off. In the context of quantum field theory, since the regularization has then the effect, in the Fourier representation, to suppress field contributions with momenta $|p| \gg \Lambda$ in the perturbative expansion, Λ is also called the cut-off.

Statistical field theory: Perturbative expansion

The Gaussian model in the critical domain

After rescaling, the Hamiltonian of the Gaussian model takes the form

$$\mathcal{H}_{\mathrm{G}}(\sigma)=\frac{1}{2}\int \mathrm{d}^d x\left[\left(\nabla_x\sigma(x)\right)^2+\alpha_0\Lambda^2\sigma^2(x)+\sum_{\ell=2}^{\ell_{\max}}\alpha_\ell\Lambda^{2-2\ell}\sigma(x)\nabla_x^{2\ell}\sigma(x)\right],$$

where α_0 is the amplitude of the only relevant term.

For $\alpha_0=0$, except for the two-point function at coinciding points, one can take the $\Lambda\to\infty$ limit. However, for $\alpha_0\neq 0$, to obtain a non-trivial universal large distance behaviour, it is also necessary to compensate the effect of the RG flow by choosing α_0 infinitesimal, that is, by taking the $\Lambda\to\infty$ limit at $r=\alpha_0\Lambda^2$ fixed (a Gaussian mass renormalization in QFT terminology). This defines the critical domain in phase transitions and leads to the fine tuning problem in particle physics.

The weakly perturbed or quasi-Gaussian model

To allow for spontaneous $\mathbb{Z}_2$ symmetry breaking and, thus, to be able to describe physics below T_c, terms have necessarily to be added to the Gaussian Hamiltonian to generate a double-well potential for constant fields.

The minimal addition, and the term relevant near the Gaussian fixed point for $d < 4$ from the RG viewpoint, is

$$\mathcal{H}_{\mathrm{G}} \mapsto \mathcal{H}(\sigma) = \mathcal{H}_{\mathrm{G}}(\sigma) + \frac{g}{4!}\Lambda^{4-d} \int \mathrm{d}^d x\, \sigma^4(x), \quad g \geq 0\,.$$

The σ^4 addition generates a shift of the critical temperature. To recover a critical theory ($T = T_c$), it is necessary to adjust the coefficient of the σ^2 term: $\alpha_0 = (\alpha_0)_c(g)$, a mass renormalization in the quantum field theory terminology, and this defines the critical Hamiltonian $\mathcal{H}_c$.

Dimensions $d > 4$

For $d > 4$, the σ^4 term, as we have shown, is an irrelevant perturbation, as the power of Λ^{4-d} confirms, which does not invalidate the universal predictions of the Gaussian model. Leading corrections to the Gaussian model are obtained by expanding in powers of the coefficient g of the σ^4 term.

In terms of $u = g\Lambda^{4-d}$, the partition function, for example, is given by

$$\mathcal{Z} = \sum_{k=0}^{\infty} \frac{(-u)^k}{(4!)^k k!} \left\langle \left(\int \mathrm{d}^d x \, \sigma^4(x) \right)^k \right\rangle_{\mathrm{G}} .$$

The Gaussian expectations values $\langle \bullet \rangle_{\mathrm{G}}$ can then be evaluated in terms of the Gaussian two-point function with the help of Wick's theorem (Feynman graph expansion).

Dimensions $d < 4$

By contrast, for any $d < 4$, the σ^4 contribution is relevant: the Gaussian fixed point is unstable and no longer governs the large distance behaviour.

This reflects in the behaviour of the perturbative expansion of the critical theory ($T = T_c$) in powers of u: it contains so-called infra-red, that is, long distance, or zero momentum in the Fourier representation, divergences.

For $d < 4$ fixed, the determination of the large distance behaviour of correlation functions requires the construction of a general renormalization group: this leads to functional equations (Wegner, Wilson) that we do not describe here, but which, in general, unfortunately cannot be solved analytically.

Renormalization group in dimension $d = 4 - \varepsilon$

However, a trick has been discovered to extend the definition of all terms of the perturbative expansion to arbitrary complex values of the dimension d in the form of meromorphic functions.

This allows studying the neighbourhood of dimension 4, replacing, in dimension $d = 4-\varepsilon$, $\varepsilon > 0$, and in the framework of a double series expansion in g and ε, the general renormalization group by a much simpler asymptotic form, valid when a non-trivial fixed point is close to the Gaussian fixed point, and studying the model analytically. Universal quantities can then be calculated as ε-expansions.

Nevertheless a numerical method has been developed, based on the field theory RG in the form of Callan–Symanzik equations, that circumvents the problem of the ε-expansion by working at large but fixed correlation length but has no small parameter and requires the additional, non-perturbative, assumption that the hierarchy of eigenoperators has not changed.

Dimensional continuation and regularization

Dimensional continuation. To define dimensional continuation, one introduces the Fourier representation of the two-point function (or propagator) $\Delta(x)$, corresponding to the Hamiltonian of the Gaussian model,

$$\Delta(x) \equiv \langle \sigma(x)\sigma(0)\rangle_{\mathrm{G}} = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \, \mathrm{e}^{-ipx} \tilde{\Delta}(p).$$

A representation of $\tilde{\Delta}(p)$ useful for dimensional continuation then is the Laplace representation (here written for the critical propagator)

$$\tilde{\Delta}(p) = \int_0^\infty \mathrm{d}s \, \rho(s\Lambda^2) \mathrm{e}^{-sp^2}, \tag{52}$$

where the property has been used that Δ is a function only of p^2. The $1/p^2$ behaviour of Δ for $p \to 0$ is then recovered if $\rho(s) \to 1$ when $s \to \infty$.

Moreover, to reduce the field integration to continuous fields and, thus, to render the perturbative expansion finite, one needs for $s \to 0$ at least $\rho(s) = O(s^q)$ with $q > (d-2)/2$.

If, in addition, one wants the expectation values of all local polynomials to be defined, one must impose to $\rho(s)$ to converge to zero faster than any power for $s \to 0$.

Examples

$$\rho(\tau) = 1 - \mathrm{e}^{-\tau} \ \Rightarrow\ \tilde{\Delta}(p) = \frac{1}{p^2} - \frac{1}{p^2 + \Lambda^2} = \frac{1}{p^2(1 + p^2/\Lambda^2)}.$$

By adding more exponentials one can reproduce the regularization by higher derivatives we have used before.

$$\rho(\tau) = \theta(\tau - 1) \ \Rightarrow\ \tilde{\Delta}(p) = \frac{1}{p^2}\,\mathrm{e}^{-p^2/\Lambda^2},$$

which regularizes monomials with any number of derivatives.

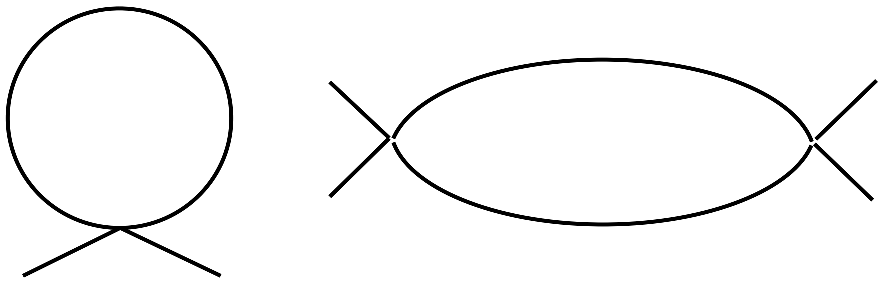

Fig. 5 Two one-loop diagrams.

A contribution to perturbation theory (represented graphically by a Feynman diagram) takes, in Fourier representation, the form of a product of propagators integrated over a subset of momenta.

With the Laplace representation, all momentum integrations become Gaussian and can be performed, resulting in explicit analytic meromorphic functions of the dimension parameter d. This can be illustrated by the two simple but useful examples of Fig. 5.

The contribution of order g to the two-point function (first diagram of Fig. 5) is proportional to

$$\begin{aligned}\Omega_d &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, \tilde{\Delta}(k) \\ &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, \int_0^\infty \mathrm{d}s\, \rho(s\Lambda^2)\mathrm{e}^{-sk^2} \\ &= \frac{1}{(4\pi)^{d/2}}\int_0^\infty \mathrm{d}s\, s^{-d/2}\rho(s\Lambda^2),\end{aligned}$$

which, in the latter form, is holomorphic for $2 < \mathrm{Re}\, d < 2(1+q)$.

Similarly, the contribution of order g^2 to the four-point function (second diagram of Fig. 5), is proportional to

$$\begin{aligned} B_d(p) &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k\, \tilde{\Delta}(k) \tilde{\Delta}(p-k) \\ &= \frac{1}{(2\pi)^d} \int \mathrm{d}^d k \int_0^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \rho(s_1\Lambda^2)\rho(s_2\Lambda^2) \mathrm{e}^{-s_1 k^2 - s_2 (p-k)^2} \\ &= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}s_1\, \mathrm{d}s_2}{(s_1+s_2)^{d/2}} \rho(s_1\Lambda^2)\rho(s_2\Lambda^2)\, \mathrm{e}^{-p^2 s_1 s_2/(s_1+s_2)}, \end{aligned}$$

which, in the latter form, is holomorphic for $2 < \mathrm{Re}\, d < 4(1+q)$.

For the theory of critical phenomena, dimensional continuation is sufficient. It allows exploring the neighbourhood of dimension four, finding fixed points and calculating universal quantities as $\varepsilon = (4-d)$-expansions.

However, for practical calculations, but then restricted to the leading large distance behaviour, a further step is extremely useful: dimensional regularization.

Dimensional regularization

It can be verified that if one decreases $\operatorname{Re} d$ enough, so that by naive power counting all momentum integrals are convergent, one can, after explicit dimensional continuation, take the infinite Λ limit. The resulting perturbative contributions become meromorphic functions with poles at dimensions at which large momentum, and low momentum in the critical theory, divergences appear.

This method of regularizing large momentum divergences is called dimensional regularization and is extensively used in quantum field theory. In the theory of critical phenomena, it has also been used to calculate universal quantities like critical exponents, as ε-expansions. For example,

$$B_d(p) = -\frac{2\pi\Gamma(d/2)}{(4\pi)^{d/2}\sin(\pi d/2)\Gamma(d-1)}p^{d-4} = \frac{1}{8\pi^2\varepsilon}(1-\varepsilon\ln p) + O(\varepsilon).$$

Perturbative renormalization group: The critical theory

The perturbative renormalization group, as it has been developed in the framework of the perturbative expansion of quantum field theory, relies on the renormalization theory.

For the σ^4 field theory it has been first formulated in space dimension $d = 4$. For critical phenomena, a minor extension is required that involves an additional expansion in powers of $\varepsilon = 4 - d$, after dimensional continuation.

We first consider the critical theory ($T = T_c$) corresponding to the Hamiltonian $\mathcal{H}_c(\sigma)$.

To formulate the renormalization theorem, one introduces a momentum scale $\mu \ll \Lambda$, called the renormalization (or physical) scale, and a parameter $g_{\rm r}$ characterizing the effective σ^4 coefficient at scale μ, called the renormalized coupling constant.

The renormalization theorem:
One can find two dimensionless functions $Z(\Lambda/\mu, g)$ and $Z_g(\Lambda/\mu, g)$ that satisfy (g and Λ/μ are the only two dimensionless combinations)

$$\Lambda^{4-d} g = \mu^{4-d} Z_g(\Lambda/\mu, g) g_{\rm r} = \mu^{4-d} g_{\rm r} + O(g^2),\ \ Z(\Lambda/\mu, g) = 1 + O(g),$$

calculable order by order in a double series expansion in powers of g and ε, such that all connected correlations functions

$$\tilde{W}_{\rm r}^{(n)}(p_i; g_{\rm r}, \mu, \Lambda) = Z^{-n/2}(g, \Lambda/\mu) \tilde{W}^{(n)}(p_i; g, \Lambda),$$

called renormalized functions, have, order by order in $g_{\rm r}$ (and ε), finite limits $\tilde{W}_{\rm r}^{(n)}(p_i; g_{\rm r}, \mu)$ when $\Lambda \to \infty$ at $p_i, \mu, g_{\rm r}$ fixed.

The renormalization constant $Z^{1/2}(\Lambda/\mu, g)$ is the ratio between the full field renormalization in presence of the σ^4 interaction and the Gaussian field renormalization $\Lambda^{(d-2)/2}$.

Universality: a first essential step

The renormalization constants Z and Z_g are defined up a multiplication by arbitrary functions of g_r.

They can be completely determined by imposing two renormalization conditions to the renormalized correlation functions, which are then independent of the specific form of the short distance regularization.

This leads to a first very important result: since initial and renormalized correlation functions are proportional, they have the same large distance behaviour. This behaviour is thus to a large extent universal since it can depend at most on only one parameter, the σ^4 coefficient g.

Perturbative limit

In addition to the limit $\tilde{W}_\mathrm{r}^{(n)}(p_i; g_\mathrm{r}, \mu)$, one defines asymptotic functions $\tilde{W}_\mathrm{as.}^{(n)}(p_i; g, \Lambda)$ and $Z_\mathrm{as.}(g, \Lambda/\mu)$ by expanding the perturbative contributions to the functions $\tilde{W}^{(n)}(p_i; g, \Lambda)$ and $Z(g, \Lambda/\mu)$, respectively, for $\Lambda \to \infty$ and keeping, order by order in g and ε, only the terms that do not go to zero.

Critical RG equations

From the relation between initial and renormalized functions and the existence of a limit $\Lambda \to \infty$, a new equation follows, obtained by differentiating the equation with respect to Λ at $\mu, g_{\rm r}$ fixed:

$$\Lambda \frac{\partial}{\partial \Lambda}\Big|_{g_{\rm r},\mu \text{ fixed}} Z^{-n/2}(g,\Lambda/\mu)\tilde{W}^{(n)}(p_i;g,\Lambda) = \Lambda \frac{\partial}{\partial \Lambda}\Big|_{g_{\rm r},\mu \text{ fixed}} \tilde{W}^{(n)}_{\rm r}(p_i;g_{\rm r},\mu,\Lambda) \to 0\,.$$

Then, introducing the asymptotic functions,

$$\Lambda \frac{\partial}{\partial \Lambda}\Big|_{g_{\rm r},\mu \text{ fixed}} Z^{-n/2}_{\rm as.}(g,\Lambda/\mu)\tilde{W}^{(n)}_{\rm as.}(p_i;g,\Lambda) = 0\,.$$

Using the chain rule, one infers

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g,\Lambda/\mu)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g,\Lambda/\mu)\right]\tilde{W}^{(n)}_{\rm as.}(p_i;g,\Lambda) = 0\,.$$

The functions β and η are defined by

$$\beta(g,\Lambda/\mu) = \Lambda \frac{\partial}{\partial\Lambda}\bigg|_{g_{\rm r},\mu} g\,, \quad \eta(g,\Lambda/\mu) = -\Lambda \frac{\partial}{\partial\Lambda}\bigg|_{g_{\rm r},\mu} \ln Z_{\rm as.}(g,\Lambda/\mu).$$

Since the functions $\tilde{W}^{(n)}_{\rm as.}$ do not depend on μ, the functions β and η cannot depend on Λ/μ and one finally obtains the RG equations (Zinn-Justin 1973):

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} + \frac{n}{2}\eta(g)\right)\tilde{W}^{(n)}_{\rm as.}(p_i;g,\Lambda) = 0\,. \tag{53}$$

From the relation between g and $g_{\rm r}$, one immediately infers that $\beta(g) = -\varepsilon g + O(g^2)$.