Calculation of the path integral

We have introduced the formal concept of path integral. The importance of the concept is that path integrals can often be calculated without referring to a limiting process, except for a global normalization.

To calculate the integral, we change variables,

$$x(\tau) \mapsto y(\tau) = x(\tau) - f(\tau),$$

where $f(\tau)$ is the solution of the variational equation

$$\delta \mathcal{S}(f) = 0 \quad \Leftrightarrow \quad \ddot{f}(\tau) = 0$$

satisfying the boundary conditions $f(0) = 0$, $f(t) = x$.

Here τ must be regarded as an index taking continuous values and $f(\tau)$ as a set of constants parametrized by the index τ. For each time τ, the change of variables is a translation and, thus, the Jacobian is 1.

The classical path $f(\tau)$ corresponds to a uniform rectilinear motion that joins the origin to the point x, in the time t:

$$f(\tau) = x\,\tau/t\,.$$

The path $y(\tau)$ then satisfies the boundary conditions

$$y(0) = y(t) = 0\,.$$

The translation leads to

$$\mathcal{S} = \frac{1}{2w_2}\left[\frac{x^2}{t} + 2\int_0^t \mathrm{d}\tau\,\dot{y}(\tau)x/t + \int_0^t \big(\dot{y}(\tau)\big)^2 \mathrm{d}\tau\right].$$

The term linear in y can be integrated explicitly and vanishes due to the boundary conditions. One infers

$$\Pi(t,\mathbf{x}) = \mathcal{N}\,\mathrm{e}^{-x^2/2w_2 t}\,.$$

The normalization $\mathcal{N}$ is given by the path integral

$$\mathcal{N} = \int [\mathrm{d}y(\tau)]\, \mathrm{e}^{-\mathcal{S}(y)}$$

with $y(0) = y(t) = 0$. It cannot be calculated in the continuum, but no longer depends on x and thus is only a function of t that is determined by probability conservation.

The path integral thus allows calculating the probability distribution in the continuum limit, by continuum methods.

Important remark.

(i) The notation $\dot{x}$ seems to imply that paths contributing to the path integral are differentiable. This is not the case. In the continuum limit, one shows that

$$\left\langle [x(\tau) - x(\tau')]^2 \right\rangle = \int \mathrm{d}x'(x - x')^2 \Pi(|\tau - \tau'|, x - x') = w_2|\tau - \tau'|.$$

Therefore, typical paths are continuous since the left-hand side vanishes for $\tau \to \tau'$.

The expectation value of the derivative squared is obtained by dividing by $(\tau - \tau')^2$ and taking the limit $|\tau - \tau'| \to 0$. The right-hand side then diverges. We conclude that typical paths of the Brownian motion are continuous but not differentiable; they only satisfy a Hlder condition of order $1/2$:

$$|x(\tau) - x(\tau')| = O(|\tau - \tau'|^{1/2}).$$

Nevertheless, the notation $\dot{x}$ is useful because the paths that yield the leading contributions to the path integral are in the vicinity of differentiable paths.

(ii) As the time-discretized expression shows, in the symbol $[\mathrm{d}x(\tau)]$ is hidden a normalization that is independent of the trajectory, but that is difficult to handle in the continuum limit. Therefore, one calculates, in general, the ratio of the path integral and a reference path integral whose value is already known.

Exercises

Exercise 1

One considers a Markovian random walk on a two-dimensional square lattice. At each time step, the walker either remains motionless with probability $1 - s$, or moves by one lattice spacing in one of the four possible directions with the same probability $s/4$, where $0 < s < 1$. At initial time $n = 0$ the walker is at the point $\mathbf{q} = 0$.

Determine the asymptotic distribution of the walker position after n steps and calculate the asymptotic distribution for $n \to \infty$. What can be said about the space symmetry of the asymptotic distribution?

Exercise 2

One considers a Markovian random walk on a cubic lattice, that is, in $\mathbb{Z}^3$. At each step the walker either remains motionless with probability $1-s$, or moves by one lattice spacing in one of the six possible directions with the same probability $s/6$, where $0 < s < 1$. At initial time $n = 0$ the walker is at the point $\mathbf{q} = 0$.

Determine the asymptotic distribution of the position of the walker after n steps when $n \to \infty$. What can be said about the space symmetry of the asymptotic distribution?

Exercise 3

A non-translation invariant evolution equation. One considers the evolution equation

$$P_n(q) = \int \mathrm{d}q' \rho(q - \sigma q') P_{n-1}(q'), \quad \sigma > 0\,.$$

Solve the problem explicitly in the large time limit.

Set up a RG formalism obtained by replacing two time steps by one. As a function of the parameter σ, determine the fixed point and the fixed point properties.

Some hints. To have an idea of the RG scheme, we iterate a first time:

$$\mathcal{T} = \int \mathrm{d}q'' \rho(q - \sigma q'')\rho(q'' - \sigma q')$$
$$= \int \mathrm{d}q'' \rho(q - \sigma^2 q' - \sigma q'')\rho(q'').$$

We then define the first transition function as $\rho_0(q)$ and the first σ parameter as σ_0. As boundary condition we have $\rho_0(q - \sigma_0 q')$.

We define

$$\rho_1(q) = \int \mathrm{d}q' \rho(q - \sigma_0 q')\rho(q')$$

and

$$\sigma_1 = \sigma_0^2$$

such that

$$\rho_0(q - \sigma_0 q') \mapsto \rho_1(q - \sigma_1 q').$$

Iterate.

Exercise 4

An example: local stability.

Consider the example

$$\rho(q) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-q^2/2}\,.$$

Starting from the RG equation

$$[\mathcal{T}\rho](q, q'') = \int \mathrm{d}q'\,\rho(q - \sigma q'')\rho(q'' - \sigma q'), \tag{9}$$

determine the value of the renormalization factor z for which the Gaussian probability distribution $\rho(q)$ is a fixed point of $\mathcal{T}_z$.

Substituting $\rho \mapsto \rho + \delta\rho$, expand equation (9) to first order in $\delta\rho$ and study the local stability of the fixed point. Show that the eigenvectors of the linear operator acting on $\delta\rho$ have the form

$$\delta\rho_p(q) = (\mathrm{d}/\mathrm{d}q)^p\,\rho(q), \quad p > 0\,.$$

Infer the corresponding eigenvalues.

Exercise 5

Random walk on a circle. To exhibit the somewhat different asymptotic properties of a random walk on compact manifolds, it is proposed to study random walk on a circle. One still assumes translation invariance. The random walk is then specified by a transition function $\rho(q-q')$, where q and q' are two angles corresponding to positions on the circle. Moreover, the function $\rho(q)$ is assumed to be periodic and continuous. Determine the asymptotic distribution of the walker position. At initial time $n=0$, the walker is at the point $q=0$.

Exercise 6

Another universality class

One considers now the transition probability $\rho(q-q')$ with

$$\rho(q)=\frac{2}{3\pi}\frac{2+q^2}{(1+q^2)^2}.$$

The initial distribution is again

$$P_0(q) = \delta(q).$$

Evaluate the asymptotic distribution $P_n(q)$ for $n \to \infty$.

Continuous phase transitions. Universality

We now discuss continuous (second order) phase transitions in statistical systems with short range interactions.

We use the terminology of ferromagnetic systems but, as a consequence of the universality property of critical phenomena, the results that are derived apply to many other transitions that are not magnetic, like the liquid–vapour, binary mixtures, superfluid Helium *etc*.... Surprisingly enough, they also apply to the statistical properties of polymers, or self-avoiding random walk on a lattice.

For these statistical systems, the correlation length, which characterizes the decay at large distance of connected correlation functions, diverges at the transition temperature: a distance scale, large with respect to the microscopic scales (range of forces, lattice spacing), is generated dynamically. A non-trivial large distance physics then appears.

As long as the correlation length remains finite, macroscopic quantities, like the mean spin, have the behaviour predicted by the central limit theorem; in the infinite volume limit, they tend toward certain values with decreasing Gaussian fluctuations.

This result can be understood in the following way: the initial microscopic degrees of freedom can be replaced by independent mean spins, attached to volumes having the correlation length as linear size. Therefore, it is natural to first study the properties of Gaussian models.

At the transition temperature T_c, and in the several phase region, the arguments are no longer valid. Nevertheless, one may wonder whether the asymptotic Gaussian measure can then be simply replaced by a perturbed Gaussian measure, that is, whether the residual correlations between mean spins can be treated perturbatively.

Such an approximation may also be called classical, perturbed Gaussian or quasi-Gaussian.

The quasi-Gaussian approximation predicts that the large distance physics has remarkably universal large distance properties at T_c, independent to a large extent of symmetries, dimension of space... Moreover, within the quasi-Gaussian approximation, universality extends to the critical domain: $|T - T_c| \ll T_c$ and small magnetic field.

A systematic study of corrections to the quasi-Gaussian approximation then allows verifying its consistency and its domain of validity. The special role of space dimension 4 emerges, which separates the higher dimensions where the approximation is justified, to lower dimensions where it cannot be valid.

For simplicity, the discussion will first be restricted to models with a discrete Ising-like $\mathbb{Z}_2$ reflection symmetry. Indeed, below T_c or in a magnetic field, models with continuous symmetries have special properties due to the presence of Goldstone modes, which require a specific analysis.

Ising-like systems: Short range two-spin interactions

Ising type ferromagnetic systems. We consider the example of a translation invariant systems of classical spins S_i on the d-dimensional lattice of points with integer coordinates, where the integer i denotes a lattice site.

The partition function in a local magnetic field H_i has the form

$$\mathcal{Z}(\mathbf{H}) = \int \left(\prod_i \rho(S_i) \mathrm{d}S_i \right) \exp\left[-\beta\, \mathcal{E}(S) + \sum_i H_i S_i \right], \tag{10}$$

where H includes a factor $\beta = 1/T$,

$\rho(S)$ is the one-site spin distribution

(for the Ising model $\rho(S) = \frac{1}{2}[\delta(S-1) + \delta(S+1)]$),

$\mathcal{E}(S)$ is the configuration energy which we choose of the form of a pair ferromagnetic interaction,

$$-\beta \mathcal{E}(S) = \sum_{i,j} V_{ij} S_i S_j \,. \tag{11}$$

We assume that the local spin distribution $\rho(S)$, like the configuration energy in zero field $\mathcal{E}(S)$, has a reflection or $\mathbb{Z}_2$ symmetry. For all $S_i \mapsto -S_i$,

$$\rho(S) = \rho(-S), \quad \mathcal{E}(-S) = \mathcal{E}(S),$$

that it decreases fast enough (or vanishes) for large spins. For $|S| > |S_0|$,

$$\rho(S) < K\,\mathrm{e}^{-b|S|^\alpha},\ b > 0\,,\ \alpha > 2\,.$$

We assume that the pair potential $V_{ij} = V_{ji}$ is a symmetric matrix with positive elements, invariant under space translations,

$$V_{ij} \equiv V(\mathbf{r}_i - \mathbf{r}_j) = V(\mathbf{r}_j - \mathbf{r}_i) \geq 0\,,$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are the vectors joining the sites i and j to the origin, and short range, which we define here as decaying at least exponentially with distance:

$$V(\mathbf{r}) \leq M\,\mathrm{e}^{-\kappa|\mathbf{r}|}, \quad \kappa > 0\,. \tag{12}$$

Mean-field theory

An early and simple description of phase transitions is based on mean field theory (MFT). It assumes that the infinite number of microscopic degrees of freedom can be replaced by a small number of macroscopic degrees of freedom and that the residual effects can be treated as perturbations.

For the class of the lattice models that we have defined, it predicts a continuous phase transition at a critical temperature where the correlation length diverges. The MFT can also be qualified as quasi-Gaussian, in the sense that it predicts the same universal properties.

The MFT can be introduced by several methods: partial summation of the high temperature expansion, variational principle, leading order of a steepest descent method.

The latter method is the most systematic and allows calculating corrections to the mean field approximation and, thus, discussing its domain of validity, which indeed is the same as for the quasi-Gaussian approximation.

Gaussian lattice model

To discuss ferromagnetic systems in the disordered phase $T > T_c$, we use here a more intuitive approach. Since we are interested only in large distance phenomena, we can replace the initial microscopic spins by classical spins σ_i, local averages of the initial spins over large volumes.

As long as the correlation length is finite and smaller than the linear size of these volumes, the spins σ_i are averages of independent spins and one expects, in the spirit of the central limit theorem, that the fluctuations of the spins σ_i are small and that their distribution, at leading order, is Gaussian.

These arguments lead to the Gaussian lattice model for which all physical quantities can be calculated exactly.

The Gaussian model. In the disordered phase, spins fluctuate around a vanishing expectation value as a consequence of the $\sigma \to -\sigma$ symmetry.

A local one-site Gaussian distribution then takes the form

$$\rho(\sigma) = \mathrm{e}^{-b_2\sigma^2/2}, \quad b_2 > 0\,.$$

A two-spin interaction of the form $\sum_{i,j} V_{ij}\sigma_i\sigma_j$, which we have assumed translation invariant, is directly quadratic.

Such a pair potential can also be considered as the first term in a small spin expansion.

The Gaussian partition function in a field can then be written as

$$\mathcal{Z}(\mathbf{H}) = \int \left(\prod_i \mathrm{d}\sigma_i\right) \exp\left[-\mathcal{H}(\sigma) + \sum_i H_i\sigma_i\right], \tag{13}$$

where

$$\mathcal{H}(\sigma) = \frac{1}{2}\sum_{i,j} \mathfrak{S}_{ij}\sigma_i\sigma_j \quad \text{with} \quad \mathfrak{S}_{ij} = b_2\delta_{ij} - 2V_{ij}\,. \tag{14}$$

The matrix $\mathfrak{S}_{ij}$ also depends only on $\mathbf{r}_i - \mathbf{r}_j$, $\mathfrak{S}_{ij} \equiv \mathfrak{S}(\mathbf{r}_i - \mathbf{r}_j)$. The model is defined only if the matrix $\mathfrak{S}_{ij}$ is strictly positive. In the thermodynamic limit, the eigenvalues of the matrix $\mathfrak{S}$ belong to a continuous spectrum given by the Fourier transform

$$\tilde{\mathfrak{S}}(\mathbf{k}) = \sum_{\mathbf{r}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\, \mathfrak{S}(\mathbf{r}) = b_2 - \sum_{\mathbf{r}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\, V(\mathbf{r}) \equiv b_2 - 2\tilde{V}(\mathbf{k}).$$

Indeed, this equation can be rewritten as the eigenvalue equation

$$\tilde{\mathfrak{S}}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}'} = \sum_{\mathbf{r}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\, \mathfrak{S}(\mathbf{r} - \mathbf{r}').$$

It is convenient to parametrize $\tilde{V}(\mathbf{k})$ as

$$\tilde{V}(\mathbf{k}) = v\, \tilde{U}(\mathbf{k}) \text{ with } \tilde{U}(\mathbf{k}) = 1 - a^2 k^2/2 + O(k^4)\,,$$

in such a way that v plays the role of the inverse temperature. One verifies that the matrix positivity condition is

$$\tfrac{1}{2} b_2 \equiv v_c > v\,.$$

The Fourier transform of the two-point function in zero field is given by

$$\tilde{W}^{(2)}(\mathbf{k}) = \tilde{\mathfrak{S}}^{-1}(\mathbf{k}) = \left[b_2 - 2v + va^2k^2 + O(k^4)\right]^{-1}.$$

The correlation length ξ diverges for $v \to v_c$, which thus corresponds to the inverse critical temperature.

For $v \to v_c$, the correlation length ξ has the universal behaviour

$$\xi \propto \frac{1}{\sqrt{v_c - v}} \propto \frac{1}{\sqrt{T - T_c}}.$$

The appearance of a large scale at the transition generates a non-trivial large distance physics. Arguments of the kind given for the random walk, lead then to expect that the corresponding universal physical properties can be described by a continuum model with a partition function given by a field integral.

Exercise 7

Phase transitions in infinite dimension: a lattice model. In infinite dimension, on a lattice every spin has an infinite number of neighbours. The model below embodies such a situation. In such a model the MFA is exact.

We consider a classical spin model with Ω Ising spins taking values ± 1.

The interaction energy, in zero field, of the model is given by

$$\beta \mathcal{E}(\mathbf{S}) = -\frac{v}{\Omega} \sum_{i,j=1}^{\Omega} S_i S_j \tag{15}$$

and, like the spin configuration, has a $\mathbb{Z}_2$ reflection symmetry. The parameter v is proportional to β, the inverse temperature. We assume $v > 0$, which favours aligned spins (*ferromagnetic interaction*).

In the model, the spatial distribution of spins plays no role and, thus, we number them simply $i = 1, \ldots, \Omega$.

The partition function in an external field H is given by

$$\mathcal{Z}(H) = \sum_{S_i=\pm 1} \exp\left[-\beta\mathcal{E}(\mathbf{S}) + H\sum_i S_i\right].$$

Because the number of terms that couple the spins is of order Ω^2, the thermodynamic limit exists only if the interaction is divided by a factor Ω.

The only physically relevant quantity is the free energy in an external magnetic field, which one intends to calculate exactly in the thermodynamic limit $\Omega \to \infty$.

(i) As an elementary preliminary, calculate the free energy and thermodynamic potential for $v = 0$ in a magnetic field.

(ii) Solve the model for all v, calculate the thermodynamic quantities in a magnetic field in the infinite Ω limit and, in particular, identify the critical value v_c of v where a phase transition occurs.

The solution is based on taking the average spin as a collective variable.

Effective statistical field theory

We have shown that the large distance properties of the simple random walk can be described by a path integral.

Heuristic arguments of the kind we have given for the random walk, lead then to expect that, even if the initial statistical system is defined in terms of random variables associated to the sites of a space lattice, and which take only a finite set of values (like, *e.g.*, the classical spins of the Ising model), the large distance properties of a system near a continuous phase transition, because the correlation length is large, a can be inferred from an effective statistical field theory in continuum space (equivalent to a local quantum field theory in imaginary time) .

Statistical field theory. An effective statistical field theory is defined in terms of a random real field $\sigma(x)$ in continuum space, $x \in \mathbb{R}^d$, and a functional measure on fields of the form $\mathrm{e}^{-\mathcal{H}(\sigma)}/\mathcal{Z}$, where $\mathcal{H}(\sigma)$ is called the Hamiltonian in statistical physics (a denomination borrowed from the statistical theory of classical gases) and the normalization $\mathcal{Z}$ is the partition function.

The partition function is given by the field integral

$$\mathcal{Z} = \int [\mathrm{d}\sigma(x)]\, \mathrm{e}^{-\mathcal{H}(\sigma)},$$

where the dependence in the temperature T is included in $\mathcal{H}(\sigma)$.

Field integrals are the generalization to d space dimensions of path integrals, and the symbol $[\mathrm{d}\sigma(x)]$ stands for summation over all fields $\sigma(x)$.

The essential condition of short range interactions in the initial statistical system translates into the property of locality of the field theory: $\mathcal{H}(\sigma)$ can be chosen as a space-integral over a linear combination of monomials in the field $\sigma(x)$ and its derivatives.

We assume also space translation and rotation invariance and, to discuss a specific case, $\mathbb{Z}_2$ reflection symmetry (like in the Ising model):

$$\mathcal{H}(\sigma) = \mathcal{H}(-\sigma).$$

In d space dimensions, a typical form then is

$$\mathcal{H}(\sigma) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \sigma(x)\right)^2 + \tfrac{1}{2} r \sigma^2(x) + \frac{g}{4!}\sigma^4(x) + \cdots\right].$$

(∇_x is the gradient vector with components $\partial/\partial x_\mu$, $\mu = 1, \ldots, d$.)

As a systematic expansion of corrections to the mean field approximation indicates, the coefficients of $\mathcal{H}(\sigma)$, like above $r, g \ldots$, are regular functions of the temperature T near the critical temperature T_c.

Functional differentiation. In what follows, we use functional differentiation. We denote functional differentiation with respect to a function $\varphi(x)$ $(x \in \mathbb{R}^d)$,

$$\frac{\delta}{\delta\varphi(x)}.$$

Functional differentiation satisfies the usual rules of differentiation (linearity and Leibnitz rule) and

$$\frac{\delta\varphi(y)}{\delta\varphi(x)} = \delta^{(d)}(x-y),$$

where $\delta^{(d)}(x)$ is the d-dimensional Dirac function.

Correlation functions

Physical observables involve field correlation functions (generalized moments of the field distribution),

$$\langle \sigma(x_1)\sigma(x_2)\dots\sigma(x_n)\rangle \equiv \frac{1}{\mathcal{Z}}\int [\mathrm{d}\sigma(x)]\sigma(x_1)\sigma(x_2)\dots\sigma(x_n)\,\mathrm{e}^{-\mathcal{H}(\sigma)}.$$

They can be derived by functional differentiation from the generating functional (generalized partition function) in an external space-dependent field $H(x)$,

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)]\,\exp\left[-\mathcal{H}(\sigma) + \int \mathrm{d}^d x\, H(x)\sigma(x)\right],$$

as

$$\langle \sigma(x_1)\sigma(x_2)\dots\sigma(x_n)\rangle = \frac{1}{\mathcal{Z}(0)}\,\frac{\delta}{\delta H(x_1)}\frac{\delta}{\delta H(x_2)}\cdots\frac{\delta}{\delta H(x_n)}\mathcal{Z}(H)\bigg|_{H=0},$$

which amounts to identifying the coefficient of the term of order H^n.

Connected correlation functions

More relevant physical observables are the connected correlation functions (generalized cumulants). The n-point function $W^{(n)}(x_1, x_2, \ldots, x_n)$ can be derived by functional differentiation from the free energy $\mathcal{W}(H) = \ln \mathcal{Z}(H)$:

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \frac{\delta}{\delta H(x_1)} \frac{\delta}{\delta H(x_2)} \cdots \frac{\delta}{\delta H(x_n)} \mathcal{W}(H) \bigg|_{H=0} .$$

Connected correlation functions have a cluster property: if in a connected n-point function one separates the points $x_1, \ldots, x_n$ into two non-empty sets, the function vanish when the distance between the two sets goes to infinity.

It is the large distance behaviour of connected correlation functions in the critical domain near T_c that may exhibit universal properties.

Translation invariance implies

$$W^{(n)}(x_1, x_2, \ldots, x_n) = W^{(n)}(x_1 + a, x_2 + a, \ldots, x_n + a) \quad \forall a\,.$$

Taking into account translation invariance, one also defines the Fourier transforms

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^{n} p_i\right) \tilde{W}^{(n)}(p_1, \ldots, p_n)$$
$$= \int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, W^{(n)}(x_1, \ldots, x_n) \exp\left(i \sum_{j=1}^{n} x_j p_j\right),$$

where, in analogy with quantum mechanics, the Fourier variables p_i are called momenta.

Thermodynamic potential. One defines a generalized thermodynamic potential $\Gamma(M)$, Legendre transform of $\mathcal{W}(H)$ (*cf.* the relation between classical Hamiltonian and Lagrangian):

$$\mathcal{W}(H) + \Gamma(M) = \int \mathrm{d}^d x \, H(x) M(x), \quad M(x) = \frac{\delta \mathcal{W}(H)}{\delta H(x)},$$

where $M(x)$ is the local magnetization. Its expansion in powers of M,

$$\Gamma(M) = \sum_n \frac{1}{n!} \int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n \, M(x_1) \dots M(x_n) \Gamma^{(n)}(x_1, \dots, x_n),$$

defines vertex functions $\Gamma^{(n)}$.

The Fourier transforms are defined by

$$(2\pi)^d \delta^{(d)}\left(\sum_{i=1}^n p_i\right) \tilde{\Gamma}^{(n)}(p_1, \dots, p_n)$$

$$= \int \mathrm{d}^d x_1 \dots \mathrm{d}^d x_n \, \Gamma^{(n)}(x_1, \dots, x_n) \exp\left(i \sum_{j=1}^n x_j p_j\right).$$

In Fourier representation, the relations between connected and vertex functions are algebraic. In particular, since

$$\int \mathrm{d}^d x' \Gamma^{(2)}(x, x') W^{(2)}(x', x'') = \delta^{(d)}(x - x'').$$

one finds

$$\tilde{\Gamma}^{(2)}(p) \tilde{W}^{(2)}(p) = 1\,.$$

Vertex functions are not directly physical but they are the functions with the simplest analytic properties. Moreover, from them connected correlation functions can be easily derived.

The Gaussian field theory

In the spirit of the central limit theorem of probabilities or the random walk, one could expect that phase transitions on large scales can be described by Gaussian or weakly perturbed Gaussian measures, since they result from an averaging over many degrees of freedom.

However, the argument assumes that the degrees of freedom are statistically independent.

This is plausible in the infinite volume limit when the correlation length is finite and the initial microscopic degrees of freedom can be replaced by effective degrees of freedom, local averages over regions of a linear size of the order of the correlation length.

The argument no longer applies at the critical temperature because the correlation length then diverges, and the problem requires a more detailed analysis.

The Gaussian model in the continuum: partition function. Let $\sigma(x)$ be a field in d-dimensional continuum space $\mathbb{R}^d$, representing an average local spin, and $H(x)$ an arbitrary local magnetic field.

We consider the Gaussian field integral, or functional integral,

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)] \exp\left[-\mathcal{H}_{\mathrm{G}}(\sigma) + \int \mathrm{d}^d x\, \sigma(x) H(x)\right], \tag{16}$$

where $\mathcal{H}_{\mathrm{G}}(\sigma)$ is the quadratic Hamiltonian

$$\mathcal{H}_{\mathrm{G}}(\sigma) = \frac{1}{2} \int \mathrm{d}^d x \left[\left(\nabla_x \sigma(x)\right)^2 + r\sigma^2(x)\right], \ r \geq 0 \,. \tag{17}$$

Comparing with the Gaussian lattice model, one verifies that $r \propto T - T_c$ and one notes that the Gaussian model can describe only the high temperature phase $T \geq T_c$.

Remarks

In the framework of quantum field theories that describe the fundamental interactions at microscopic scale, the Gaussian model corresponds to a free scalar field theory in imaginary time.

The form (17), quadratic in the fields, is then called Euclidean action, or action in imaginary time, and the parameter

$$m = \sqrt{r}$$

is the mass of the quantum particle associated with the field σ.

As in lattice models, when this seems necessary, we define the infinite volume or thermodynamic limit as the limit of a cube with periodic boundary conditions.

Maximum of the integrand and two-point function

The calculation of a Gaussian field integral is a simple generalization of the calculation of the Gaussian path integral. One first looks for a maximum of the integrand and thus the minimum of

$$\mathcal{H}_{\mathrm{G}}(\sigma, H) = \mathcal{H}_{\mathrm{G}}(\sigma) - \int \mathrm{d}^d x \, \sigma(x) H(x). \tag{18}$$

One sets

$$\sigma(x) = \sigma_c(x) + \varsigma(x) \tag{19}$$

and expands in ς. The field $\sigma_c(x)$ at the minimum, is determined by the condition that the term linear in ς vanishes:

$$-\int \mathrm{d}^d x \left[\nabla_x \sigma_c(x) \cdot \nabla_x \varsigma(x) + m^2 \sigma_c(x) \varsigma(x)\right] + \int \mathrm{d}^d x \, \varsigma(x) H(x) = 0 \, .$$

One integrates the term linear in $\nabla_x \varsigma$ by parts. Then, because the integrated terms cancel due to periodic boundary conditions,

$$\int \mathrm{d}^d x \, \nabla_x \sigma_c(x) \cdot \nabla_x \varsigma(x) = -\int \mathrm{d}^d x \, \varsigma(x) \nabla_x^2 \sigma_c(x).$$

One finds the equation

$$(-\nabla_x^2 + m^2)\sigma_c(x) = H(x),$$

where ∇_x^2 is the Laplacian in d dimensions. The solution can be written as

$$\sigma_c(x) = \int \mathrm{d}^d x\, \Delta(x-y) H(y),$$

where Δ satisfies ($\delta^{(d)}$ is the Dirac distribution in d dimensions)

$$(-\nabla_x^2 + m^2)\Delta(x) = \delta^{(d)}(x),$$

as one verifies by acting with $-\nabla_x^2 + m^2$ on σ_c.

The equation can be solved by Fourier transformation. In the infinite volume limit, one finds

$$\Delta(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot x}}{k^2 + m^2} \,,$$

as one verifies by acting with $-\nabla_x^2 + m^2$ on $\Delta(x)$ since

$$\int \mathrm{d}^d k \; \mathrm{e}^{-ik\cdot x} = (2\pi)^d \delta^{(d)}(x).$$

After an integration by parts, the Hamiltonian for $\sigma = \sigma_c$ then becomes

$$\begin{aligned} \mathcal{H}_{\mathrm{G}}(\sigma_c, H) &= \int \mathrm{d}^d x \, \sigma_c(x) \left[-\tfrac{1}{2}\nabla_x^2 + \tfrac{1}{2}m^2 - H(x)\right] \sigma_c(x) \\ &= -\frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, H(x)\Delta(x-y)H(y). \end{aligned} \tag{20}$$

Gaussian integration

One now performs the change of variables $\sigma(x) \mapsto \varsigma(x) = \sigma(x) - \sigma_c(x)$. The initial Hamiltonian becomes

$$\mathcal{H}_{\mathrm{G}}(\sigma, H) = \mathcal{H}_{\mathrm{G}}(\sigma_c, H) + \mathcal{H}_{\mathrm{G}}(\varsigma).$$

Thus,

$$\mathcal{Z}(H) = \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\sigma_c, H)} \int [\mathrm{d}\varsigma(x)]\, \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\varsigma)}\,.$$

The remaining Gaussian integration over $\varsigma(x)$ yields a normalization,

$$\mathcal{Z}(0) = \int [\mathrm{d}\varsigma(x)]\, \mathrm{e}^{-\mathcal{H}_{\mathrm{G}}(\varsigma)},$$

independent of H, and that can be explicitly evaluated only by replacing the continuum by a lattice.

One concludes

$$\mathcal{Z}(H) = \mathcal{Z}(0) \exp\left[\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^d y\, H(x)\Delta(x-y)H(y)\right].$$

Differentiating twice with respect to $H(x)$ and setting $H=0$, one obtains the two-point function

$$\langle \sigma(x_1)\sigma(x_2)\rangle = \Delta(x_1 - x_2).$$

In the case of a centred Gaussian measure, all correlation functions or moments can then be expressed in terms of the two-point function, or second moment, with the help of Wick's theorem. Here,

$$\langle \sigma(x_1)\dots\sigma(x_{2n}\rangle = \sum_{\substack{\text{all pairings}\\ \text{of } \{1,2,\dots,2n\}}} \Delta(x_{i_1} - x_{i_2})\dots\Delta(x_{i_{2n-1}} - x_{i_{2n}}).$$

A few results. The generating functional of connected correlation functions thus is

$$\mathcal{W}(H) = \ln \mathcal{Z}(H) = \mathcal{W}(0) + \frac{1}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, H(x) \Delta(x-y) H(y).$$

Differentiating $\mathcal{W}(H)$ twice with respect to $H(x)$, one verifies that $\Delta(x-y)$ is also the connected Gaussian two-point function,

$$W^{(2)}(x,y) = \Delta(x-y) = \langle \sigma(x)\sigma(y) \rangle \, .$$

It has an Ornstein–Zernike or free field form.

The two-point function is the unique connected correlation function.

In a uniform field, the free energy density becomes

$$\begin{aligned} W(H) &= \left(\mathcal{W}(H) - W(0)\right)/\text{volume} \\ &= \tfrac{1}{2} H^2 \int \mathrm{d}^d x \, \Delta(x) = \tfrac{1}{2} H^2 \tilde{\Delta}(0) = \tfrac{1}{2} H^2 / m^2 . \end{aligned}$$

The equation of state, relation between magnetic field H, magnetization M and temperature is given by

$$M = \frac{\partial W}{\partial H} = \frac{H}{r} \propto \frac{H}{T - T_c} \,.$$

The magnetic susceptibility follows

$$\chi = \frac{\partial M}{\partial H} = \frac{1}{r} \propto \frac{1}{T - T_c} \,.$$

In general, one defines an exponent γ that characterizes the divergence of χ at T_c. Here $\gamma = 1$.

The thermodynamic potential density, Legendre transform of $W(H)$ then is

$$\mathcal{G}(M) = \tfrac{1}{2} m^2 M^2 .$$

Explicit calculation of the two-point function

First, at T_c,

$$W^{(2)}(x) = \Delta(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot x}}{k^2} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-ik\cdot x} \int_0^\infty \mathrm{d}t \; \mathrm{e}^{-tk^2} \quad .$$

Performing the Gaussian integral over k, one finds

$$\Delta(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t}{t^{d/2}} \, \mathrm{e}^{-x^2/4t} \, .$$

After the change of variables $u = x^2/4t$, the integration over u yields

$$\Delta(x) = \frac{2^{d-2}}{(4\pi)^{d/2}} \Gamma(d/2 - 1) \frac{1}{x^{d-2}} \, . \qquad (21)$$

At T_c, the two-point function is not defined for $d = 2$. For $d > 2$, , one finds an algebraic decay of the two-point function at large distance.

In general one defines (quite generally, one then proves $\eta \geq 0$)

$$W^{(2)}(x) \underset{x\to\infty}{\propto} \frac{1}{x^{d-2+\eta}}.$$

In the Gaussian model one finds the exponent value $\eta = 0$.

For the function $1/(k^2 + m^2)$ the strategy is the same. One then finds

$$\Delta(x) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\mathrm{d}t}{t^{d/2}} \,\mathrm{e}^{-x^2/4t - m^2 t} = \frac{2}{(4\pi)^{d/2}} \left(\frac{2m}{x}\right)^{d/2-1} K_{1-d/2}(mx),$$

where $K_\nu(z)$ is a Bessel function of third kind. For $z \to +\infty$, $K_\nu(z)$ can be evaluated by the steepest descent method. One infers

$$\Delta(x) \underset{x\to\infty}{\sim} \frac{1}{2m} \left(\frac{m}{2\pi}\right)^{(d-1)/2} \frac{\mathrm{e}^{-mx}}{x^{(d-1)/2}}.$$

The correlation length $\xi = 1/m$ characterizes the exponential decay of the two-point function. Near T_c it behaves like $\xi \propto (T - T_c)^{-\nu}$ with the exponent value $\nu = \frac{1}{2}$.

Class of fields contributing to the field integral. To get an idea of the class of typical fields that contribute to the field integral, one can evaluate the two-point function in the limit of coinciding points:

$$\langle\sigma(\mathbf{x})\sigma(\mathbf{y})\rangle \underset{|\mathbf{x}-\mathbf{y}|\to 0}{\sim} \Delta(\mathbf{x}-\mathbf{y}, m=0) = \frac{2^{d-2}}{(4\pi)^{d/2}}\Gamma(d/2-1)\frac{1}{|\mathbf{x}-\mathbf{y}|^{d-2}}\,.$$

One notices that the fields $\sigma(\mathbf{x})$ contributing to the field integrals are so singular that the expectation value of $\sigma^2(\mathbf{x})$ diverges for $d > 1$, and with a rate that increases with the dimension of space d. This singularity of the Gaussian measure is the source of new difficulties.

For $d = 2$, the short distance behaviour takes the form

$$\langle\sigma(\mathbf{x})\sigma(\mathbf{y})\rangle \underset{|\mathbf{x}-\mathbf{y}|\to 0}{\sim} -\frac{1}{2\pi}\ln(m|\mathbf{x}-\mathbf{y}|).$$

Quasi-Gaussian or classical approximation

Below the transition point, the Gaussian model is clearly no longer valid since the quadratic form in the Hamiltonian is not positive and thus the Gaussian integral is not defined.

However, even in the framework of the central limit theorem, the Gaussian distribution is only asymptotic. The analysis of the Gaussian model shows that below the transition point, corrections to the Gaussian distribution, that is, terms of higher degree in the effective field distribution, even if their amplitude is small, can no longer be neglected.

Quasi-Gaussian approximation. Since the field integral then is no longer Gaussian, it cannot be calculated exactly. But since the Hamiltonian remains formally analytic, the integral over the fields can be evaluated by the steepest descent method.

Moreover, if one assumes that the fluctuations around the saddle point vary slowly with H, one can neglect the contributions coming from integrating out the fluctuations around the saddle point and approximate the free energy by value of the Hamiltonian at the saddle point, an approximation which one can call quasi-Gaussian or classical.

Such an assumption implies, in particular, that the fields $\sigma(x)$ are the sum of an average value $M(x)$ and a weakly correlated fluctuating part.

This assumption goes beyond an idea of central limit theorem in the sense that the average value $M(x)$ is no longer related only to the distribution in each site but also results from the interactions.

One can show that the quasi-Gaussian approximation reproduces, at leading order, the universal results of the lattice model in infinite dimension. However, unlike the model in infinite dimension, it allows also studying the behaviour of correlation functions at the transition.

Effective model. To go beyond the Gaussian model, we thus consider a more general one-site local distribution. In the continuum limit, this corresponds to adding to the Hamiltonian $\mathcal{H}_{\mathrm{G}}$ a local function of the form

$$\int \mathrm{d}^d x\, B\big(\sigma^2(x)/2\big),$$

$B(\sigma^2/2)$ having the properties of the thermodynamic potential of a one-site model: it is a convex analytic function.

We parametrize its expansion at $\sigma = 0$ in the form

$$B(\Sigma) = \sum_{p=1} \frac{2^p}{2p!} b_{2p} \Sigma^p \,, \quad b_2 > 0 \,. \tag{22}$$

One verifies that the existence of a continuous transition implies $b_4 > 0$.

The generating functional of correlation functions can then be written as

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma(x)] \exp\left[-\mathcal{H}(\sigma) + \int \mathrm{d}^d x\, H(x)\sigma(x)\right] \tag{23}$$

with

$$\mathcal{H}(\sigma) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\big(\nabla_x \sigma(x)\big)^2 + \tfrac{1}{2} r\sigma^2(x) + B\big(\sigma^2(x)/2\big)\right]. \tag{24}$$

Steepest descent method. The maximum of the integrand in the field integral (23) is given by a solution of the saddle point field equation

$$\frac{\delta \mathcal{H}}{\delta \sigma(x)} = H(x). \tag{25}$$

At leading order, approximating the field integral by its saddle point value, one finds

$$\mathcal{W}(H) = -\mathcal{H}(\sigma) + \int \mathrm{d}^d x\, \sigma(x) H(x),$$

where σ is a function of H through (25).

Together, these equations show that $\mathcal{W}(H)$ is the Legendre transform of $\mathcal{H}(\sigma)$. As a consequence, the thermodynamic potential $\Gamma(M)$, Legendre transform of $\mathcal{W}(H)$, is simply

$$\Gamma(M) = \mathcal{H}(M). \tag{26}$$

In a uniform magnetic field H, the thermodynamic potential density is

$$\mathcal{G}(M) = \frac{\Gamma(M)}{\text{volume}} = \tfrac{1}{2} r M^2 + B(M^2/2). \tag{27}$$

The equation of state follows:

$$H = \frac{\partial \mathcal{G}}{\partial M} = rM + MB'(M^2/2). \tag{28}$$

For $H = 0$, the magnetization is solution of (B is a convex function)

$$rM + MB'(M^2/2) = M(r + b_2) + \tfrac{1}{6}b_4 M^3 + O(M^5) = 0\,.$$

For $r > r_c = -b_2$, the minimum of the thermodynamic potential corresponds to the symmetric solution $M = 0$.

For $r < r_c = -b_2$, it corresponds to a non-vanishing value, the spontaneous magnetization (we have assumed $b_4 > 0$), which for $0 < r_c - r \ll 1$ is given by

$$M \sim \pm\sqrt{6(r_c - r)/b_4}\,. \tag{29}$$

If one defines in general a critical exponent β by $M \propto (T_c - T)^\beta$, one finds the mean field value $\beta = \frac{1}{2}$.

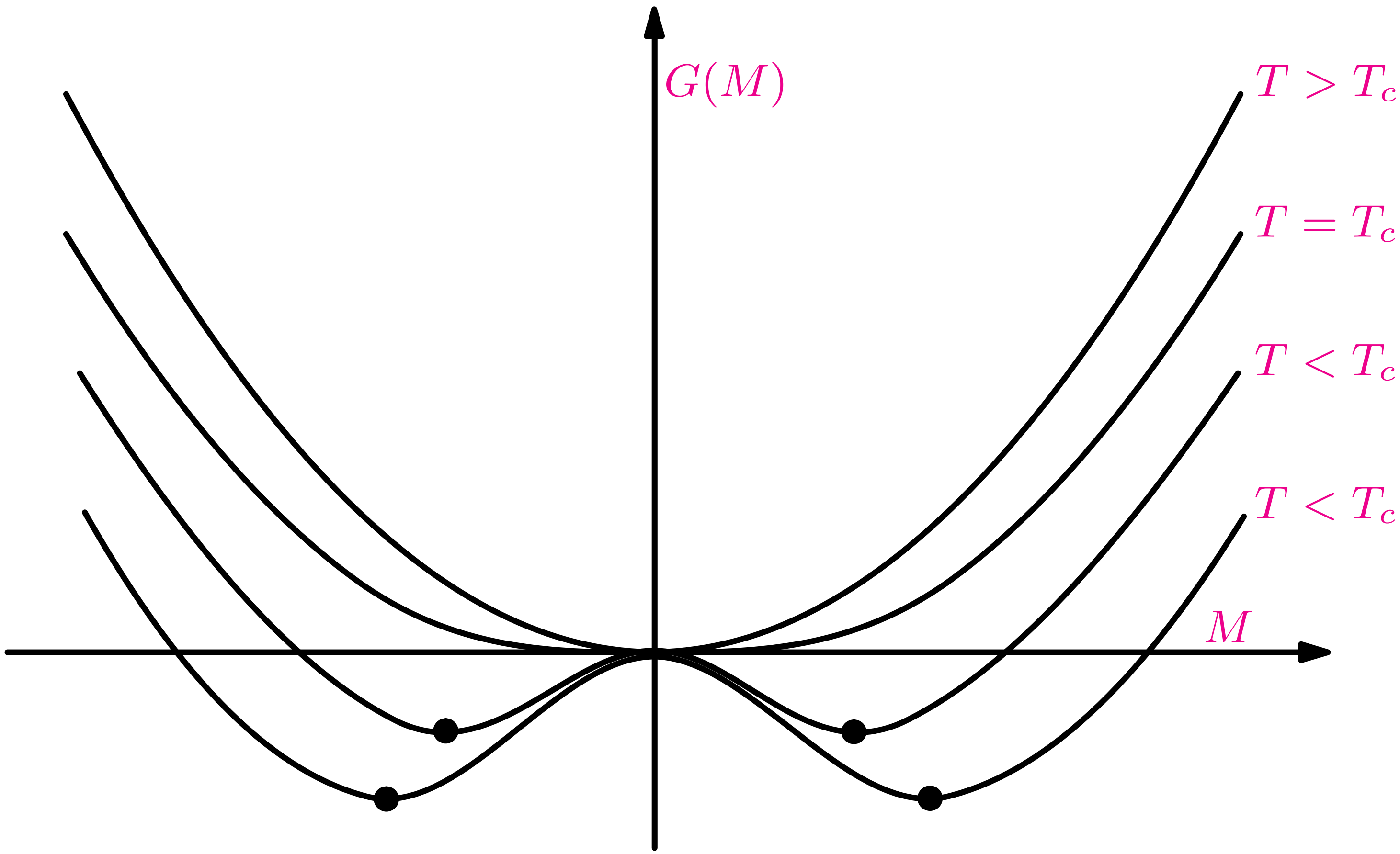

Fig. 2 Thermodynamic potential: second order phase transition.

Quasi-Gaussian approximation: The two-point function

Divergence of the correlation length and continuous transition. A continuous transition is characterized by the property

$$\left.\frac{\partial^2 \mathcal{G}}{(\partial M)^2}\right|_{M=0} = 0 \tag{30}$$

and, thus, by the divergence of the magnetic susceptibility $\chi = \partial^2 W/(\partial H)^2$ in zero field. Moreover,

$$\frac{\partial W(H)}{\partial H} = \left.\frac{\delta \mathcal{W}}{\delta H(x)}\right|_{H(x)=H} .$$

The second derivative is, thus, related to the connected two-point function:

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d y \left.\frac{\delta^2 \mathcal{W}}{\delta H(x)\delta H(y)}\right|_{H(x)=H} = \int \mathrm{d}^d y\, W^{(2)}(x,y).$$

Translation invariance in a uniform field implies

$$W^{(2)}(x,y) = W^{(2)}(x-y).$$

Thus,

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d x\, W^{(2)}(x).$$

We now introduce the Fourier transforms of the connected and vertex functions

$$\widetilde{W}^{(2)}(k) = \int \mathrm{d}^d x \ \mathrm{e}^{ik\cdot x}\, W^{(2)}(x), \quad \tilde{\Gamma}^{(2)}(k) = \int \mathrm{d}^d x \ \mathrm{e}^{ik\cdot x}\, \Gamma^{(2)}(x).$$

Then,

$$\frac{\partial^2 W(H)}{(\partial H)^2} = \int \mathrm{d}^d x\, W^{(2)}(x) = \widetilde{W}^{(2)}(k=0) = 1/\tilde{\Gamma}^{(2)}(k=0).$$

The integral $\int \mathrm{d}^d x\, W^{(2)}(\mathbf{x})$ diverges only if the correlation length diverges. The condition of continuous transition thus implies the divergence of the correlation length for vanishing magnetization.

Two-point function at fixed magnetization. More generally, from $\Gamma(M) = \mathcal{H}(M)$ one infers the relation between local magnetic field and magnetization

$$H(x) = \frac{\delta\Gamma}{\delta M(x)} = -\nabla_x^2 M(x) + rM(x) + M(x)B'\big(M^2(x)/2\big). \tag{31}$$

By differentiating the equation with respect to $M(y)$ and setting $M(x) = M$, one obtains the two-point vertex function at fixed magnetization

$$\begin{aligned}\Gamma^{(2)}(x-y) &\equiv \left.\frac{\delta^2\Gamma}{\delta M(x)\delta M(y)}\right|_{M(x)=M} \\ &= [-\nabla_x^2 + r + B'(M^2/2) + M^2 B''(M^2/2)]\delta^{(d)}(x-y).\end{aligned}$$

Its Fourier transform is given by

$$\tilde{\Gamma}^{(2)}(k) = k^2 + r + B'(M^2/2) + M^2 B''(M^2/2). \tag{32}$$

The Fourier transform of the connected two-point function follows:

$$\widetilde{W}^{(2)}(k) = 1/\tilde{\Gamma}^{(2)}(k) = \big(k^2 + r + B'(M^2/2) + M^2 B''(M^2/2)\big)^{-1}.$$

The correlation length above and below T_c

In zero field, above T_c, the magnetization vanishes and one recovers the form of the Gaussian model

$$\widetilde{W}^{(2)}(k) = \left(r - r_c + k^2\right)^{-1},$$

where $r_c = -b_2$. If the transition is second order, the expression remains valid up to $r = r_c$ ($T = T_c$) where the correlation length diverges. In particular, one recovers the Gaussian or classical values of the exponents $\eta = 0$ and $\nu = 1/2$ with

$$\xi \equiv \xi_+ \sim (r - r_c)^{-1/2} \propto (T - T_c)^{-1/2} \quad \text{for} \quad T > T_c\,.$$

More generally, for $|r - r_c|$, $|k|$, $M \ll 1$ (which also implies a weak magnetic field) one finds

$$\widetilde{W}^{(2)}(k) \sim \left(k^2 + r - r_c + \tfrac{1}{2} b_4 M^2 + O(M^4)\right)^{-1}. \tag{33}$$

The correlation function retains an Ornstein–Zernike or free field form. The correlation length for $M \neq 0$ follows:

$$\xi^{-2} = r - r_c + \tfrac{1}{2} b_4 M^2 . \tag{34}$$

In zero magnetic field, using below T_c the expression (29) of the spontaneous magnetization, one finds

$$M^2 = 6(r - r_c)/b_4 \ \Rightarrow \ \xi \equiv \xi_- \sim [2(r_c - r)]^{-1/2} \propto (T_c - T)^{-1/2} \text{ for } \quad T < T_c \, . \tag{35}$$

Introducing also quite generally a correlation length exponent ν' for $T \to T_{c-}$, and defining the critical amplitudes $f_\pm$ for $|T - T_c| \to 0$ by

$$\xi_+ \sim f_+ \, (T - T_c)^{-\nu} \, , \quad \xi_- \sim f_- \, (T_c - T)^{-\nu'} \, ,$$

one infers the quasi-Gaussian value of the exponent $\nu' = \nu = \frac{1}{2}$ and the universal amplitude ratio

$$f_+ / f_- = \sqrt{2} \, .$$

Notice that sometimes the correlation length is defined in terms of the second moment ξ_1^2 of $W^{(2)}(x)$ which is proportional to ξ^2, and thus has the same universal properties

$$\tilde{\Gamma}^{(2)}(k) = \left[\widetilde{W}^{(2)}(k)\right]^{-1} \sim \tilde{\Gamma}^{(2)}(0)\left(1 + k^2\xi_1^2 + O(k^4)\right). \tag{36}$$

Another universal amplitude. If for $r = r_c$, $H \to 0$, one sets

$$\chi \sim C^c/H^{2/3}, \;\Rightarrow\; 3C^c = (6/b_4)^{1/3}$$

and, in zero field,

$$M \sim M_0(r - r_c)^{1/2} \;\Rightarrow\; M_0^2 = 12/b_4\,.$$

Then, the combination

$$R_\chi = C^+ M_0^2 (3C^c)^{-3} = 1\,,$$

is universal.

Continuous symmetries and Goldstone modes

If the initial spin variables S_i are N-component vectors and if the interaction and the local spin distribution have a continuous symmetry (corresponding to a compact Lie group), many results obtained so far within the framework of the quasi-Gaussian approximation remain unchanged.

However, the appearance of several types of correlation functions and correlation lengths when the magnetization is different from zero, induces some new properties. In particular, in zero field, at any temperature below T_c some correlation lengths, associated with modes called Goldstone modes, diverge.

We verify these properties within the quasi-Gaussian approximation in the case of models having an orthogonal $O(N)$ symmetry ($N > 1$), that is, invariant under the group of space rotations–reflections in N dimensions acting on the N-component field $\boldsymbol{\sigma}$.

Quasi-Gaussian approximation

In this more general situation, the thermodynamic potential has a structure analogous to expression (26), except that the local magnetization $\mathbf{M}(x)$ now is an N-component vector and the thermodynamic potential is invariant under orthogonal transformations acting on the vector $\mathbf{M}(x)$.

The $O(N)$ invariance implies that the thermodynamic potential can be expressed in terms of scalar products and, thus, can be written as

$$\Gamma(\mathbf{M}) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\nabla_x \mathbf{M}(x)\right)^2 + \tfrac{1}{2} r \mathbf{M}^2(x) + B\left(\mathbf{M}^2(x)/2\right)\right] \tag{37}$$

where the function $B(\Sigma)$ is identical to the function (22):

$$B(\Sigma) = b_2 \Sigma + \frac{b_4}{6}\Sigma^2 + \cdots \quad \text{with} \quad b_2 > 0\,,$$

where the assumption of a continuous transition again implies $b_4 > 0$.

Equation of state

In a uniform field, the thermodynamic potential density then reads

$$\mathcal{G}(\mathbf{M}) = \tfrac{1}{2} r \mathbf{M}^2 + B(\mathbf{M}^2/2).$$

The equation of state follows:

$$H_\alpha = \frac{\partial \mathcal{G}(\mathbf{M})}{\partial M_\alpha} = M_\alpha \bigl(r + B'(M^2/2)\bigr).$$

Taking the modulus of both members, one obtains a form analogous to the $\mathbb{Z}_2$ Ising-like situation:

$$H = M\bigl(r + B'(M^2/2)\bigr), \tag{38}$$

where H, M now are the lengths of the vectors $\mathbf{H}$, $\mathbf{M}$.

The existence of a continuous phase transition at $r_\mathrm{c} = -b_2$ and the universal properties of the equation of state follow then from arguments identical to those already presented in the case of the discrete reflection symmetry $\mathbb{Z}_2$.

Two-point correlation function

Differentiating expression (37), one finds ($1 \le \alpha \le N$)

$$\frac{\delta\Gamma(\mathbf{M})}{\delta M_\alpha(x)} = \left(-\nabla_x^2 + r\right) M_\alpha(x) + M_\alpha(x) B'\big(\mathbf{M}^2(x)/2\big), \tag{39}$$

where $M_\alpha(x)$ are the components of the local magnetization vector $\mathbf{M}(x)$.

The functional derivative with respect to $M_\beta(y)$ of equation (39) yields, in the limit $\mathbf{M}$ uniform, the two-point vertex function,

$$\Gamma^{(2)}_{\alpha\beta}(x-y) \equiv \left.\frac{\delta^2\Gamma(\mathbf{M})}{\delta M_\alpha(x)\delta M_\beta(y)}\right|_{\mathbf{M}(x)=\mathbf{M}}$$
$$= \left[\left(-\nabla_x^2 + r + B'(M^2/2)\right)\delta_{\alpha\beta} + M_\alpha M_\beta B''(M^2/2)\right]\delta^{(d)}(x-y).$$

The Fourier components of the two-point vertex function then are

$$\tilde{\Gamma}^{(2)}_{\alpha\beta}(\mathbf{k}) = \left[\mathbf{k}^2 + r + B'(M^2/2)\right]\delta_{\alpha\beta} + M_\alpha M_\beta B''(M^2/2)\,. \tag{40}$$

The function $\tilde{\Gamma}^{(2)}_{\alpha\beta}(\mathbf{k})$ remains an $N \times N$ matrix in the N-vector space. Its inverse in the sense of matrices is the connected correlation function $\widetilde{W}^{(2)}_{\alpha\beta}(\mathbf{k})$.

We introduce a unit vector along the direction of the magnetization:

$$\mathbf{M} = M\mathbf{u} \quad \text{with} \quad \mathbf{u}^2 = 1\,.$$

The matrix $\tilde{\Gamma}^{(2)}_{\alpha\beta}$ has two eigenspaces corresponding to the vector $\mathbf{u}$ and the vectors orthogonal to $\mathbf{u}$. The function (40) can then be decomposed into transverse and longitudinal parts:

$$\tilde{\Gamma}^{(2)}_{\alpha\beta} = u_\alpha u_\beta \tilde{\Gamma}^{(2)}_{\mathrm{L}} + (\delta_{\alpha\beta} - u_\alpha u_\beta)\,\tilde{\Gamma}^{(2)}_{\mathrm{T}}\,,$$

where $\tilde{\Gamma}^{(2)}_{\mathrm{L}}$, $\tilde{\Gamma}^{(2)}_{\mathrm{T}}$ are the two eigenvalues, respectively, given by

$$\tilde{\Gamma}^{(2)}_{\mathrm{L}}(\mathbf{k}) = \mathbf{k}^2 + r + B'(M^2/2) + M^2 B''(M^2/2), \tag{41a}$$

$$\tilde{\Gamma}^{(2)}_{\mathrm{T}}(\mathbf{k}) = \mathbf{k}^2 + r + B'(M^2/2). \tag{41b}$$

The expressions $(41a)$ and (33) are similar. Using the equation of state $H/M = r + B'(M^2/2)$, one can rewrite the second eigenvalue as

$$\tilde{\Gamma}^{(2)}_{\mathrm{T}}(\mathbf{k}) = \mathbf{k}^2 + H/M\,.$$

Since $\tilde{\Gamma}^{(2)}_{\mathrm{L}}$, $\tilde{\Gamma}^{(2)}_{\mathrm{T}}$ are the eigenvalues of the matrix $\tilde{\Gamma}^{(2)}$, the matrix of connected functions has the inverse eigenvalues:

$$\widetilde{W}^{(2)}_{\mathrm{L}}(\mathbf{k}) = \left[\tilde{\Gamma}^{(2)}_{\mathrm{L}}(\mathbf{k})\right]^{-1}, \quad \widetilde{W}^{(2)}_{\mathrm{T}}(\mathbf{k}) = \left[\tilde{\Gamma}^{(2)}_{\mathrm{T}}(\mathbf{k})\right]^{-1}.$$

Goldstone modes

At any temperature $T < T_{\mathrm{c}}$, the ratio H/M vanishes for $H \to 0$ because M becomes the non-vanishing spontaneous magnetization. The explicit form of $\widetilde{W}^{(2)}_{\mathrm{T}}(\mathbf{k})$ shows that in zero field in the ordered phase, the transverse two-point correlation function diverges like $1/k^2$ for $k \to 0$, a result that implies the existence of $(N-1)$ Goldstone modes with infinite correlation length (massless particles in the QFT terminology).

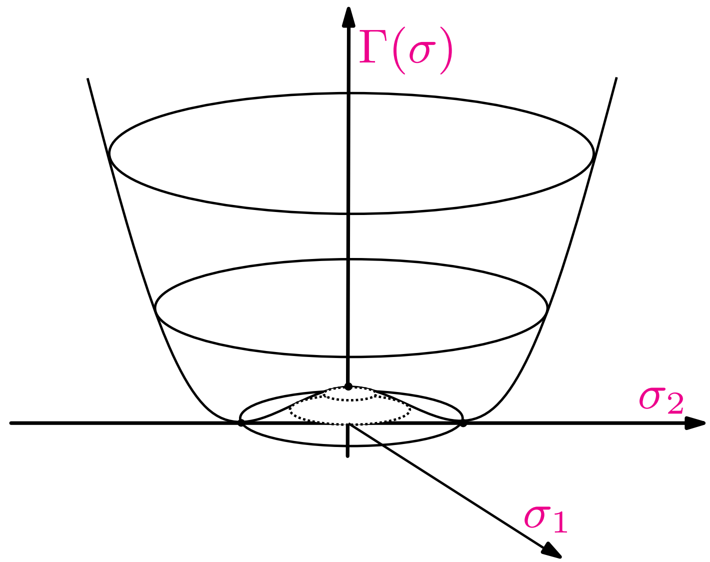

Fig. 3 An $O(2)$ symmetric potential $\Gamma(\sigma)$ with degenerate minima.

Exercise 8

A model with $O(3)$ orthogonal symmetry in infinite dimension.

One generalizes the exercise 7 to a model where the spins $\mathbf{S}$ are three-component vectors belonging to the sphere S_2, $\mathbf{S}^2 = 1$, with an interaction energy of the $O(3)$ invariant form

$$-\beta\mathcal{E}(\mathbf{S}) = \frac{v}{\Omega}\sum_{i,j=1}^{\Omega}\mathbf{S}_i \cdot \mathbf{S}_j\,, \quad v > 0\,.$$

(i) Calculate first partition function in a field for $v = 0$, the corresponding free energy and thermodynamic potential. Expand the thermodynamic potential up to fourth order in the three-component magnetization vector.

(ii) It is then suggested to introduce the distribution of the mean-spin, a three-component vector. Calculate the thermodynamic quantities, in particular, determine the critical value v_c of v (the inverse critical temperature).