RENORMALIZATION GROUP: AN INTRODUCTION

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The renormalization group (RG) has played a crucial role in 20th century physics in two apparently unrelated domains: the theory of fundamental interactions at the microscopic scale or particle physics and the theory of continuous macroscopic phase transitions.

In particle physics, the necessity of renormalization to cancel infinities that appear in a straightforward interpretation of quantum field theory (QFT), and of the freedom of defining the parameters of the renormalized or physical theory at different momentum scales led to a first form of RG.

In the statistical physics of phase transitions, a more general renormalization group, based on a recursive averaging over short distance degrees of freedom, was later introduced to explain universal properties within various classes of continuous phase transitions.

The renormalization group of quantum field theory can now be understood as the asymptotic form of the general renormalization group in some neighbourhood of the Gaussian fixed point.

In these lectures, we first illustrate the notion of universality with the elementary example of the translation invariant random walk, a problem directly related to the central limit theorem of probabilities.

We revisit the problem with RG inspired methods, introducing in this way the RG terminology. We recover that the large time and space behaviour is universal and defines a continuum limit that can be described by a path integral.

Then, we argue that similarly large distance properties of statistical models near a continuous phase transition can be described by statistical field theories.

We explain the perturbative renormalization group. We review a few important applications like the proof of scaling laws and the determination of singularities of thermodynamic functions at the phase transition.

For an elementary introduction to the renormalization group in the spirit of these lectures, *cf.*, for example,

J. Zinn-Justin, *Phase transitions and renormalization group*, Oxford Univ. Press (Oxford 2007) and Peking Univ. Press (Beijing 2017),

adapted from the French edition *Transitions de phase et groupe de renormalisation.* EDP Sciences/CNRS Editions, Les Ulis 2005,

including the functional renormalization group in Chapter 16.

In www.scholarpedia.org, see

Jean Zinn-Justin (2010), *Critical Phenomena: field theoretical approach*, Scholarpedia, 5(5):8346.

More advanced material can be found in

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002),

including critical dynamics in Chapter 36.

Universality and continuum limit: The random walk

The universality of large scale behaviour and, correspondingly, the existence of a macroscopic continuum limit, emerge as collective properties of systems involving a large number of random variables whose individual distribution is sufficiently localized.

These properties, as well as the appearance of an asymptotic Gaussian distribution when the random variables are statistically independent, are illustrated here with the simple example of the random walk.

The random walk with discrete time steps: a renormalization group viewpoint. Inspired by renormalization group (RG) ideas, we introduce transformations, acting on the transition probability, which decrease the number of random time steps. We show that Gaussian distributions are attractive fixed points for these transformations. The continuum asymptotic limit with universal scaling properties is then recovered.

The properties of the continuum limit can then be described by a path integral.

Random walk invariant under time translation

We consider a stochastic process, a random walk, in discrete times, first on the real axis and then, briefly, on the lattice of points with integer coordinates.

The random walk is specified by

an initial probability distribution $P_0(q)$ (q being a position) at time $n = 0$,

a density of transition probability $\rho(q, q') \geq 0$ from the point q' to the point q, which we assume independent of the (integer) time n.

These conditions define a Markov chain, a Markovian process, in the sense that the displacement at time n depends only on the position at time n, but not on the positions at prior times, homogeneous or stationary, that is, invariant under time translation, up to the boundary condition.

Random walk in continuum space

Probability conservation implies

$$\int \mathrm{d}q\, \rho(q, q') = 1\,. \tag{1}$$

The probability distribution $P_n(q)$ for a walker to be at point q at time n satisfies the evolution equation

$$P_{n+1}(q) = \int \mathrm{d}q'\, \rho(q, q')P_n(q'), \quad \int \mathrm{d}q\, P_0(q) = 1\,.$$

Equation (1) then implies $\int \mathrm{d}q\, P_n(q) = 1$.

To slightly simplify the analysis, we take as initial distribution $P_0(q) = \delta(q)$, where δ is Dirac's distribution (the walker at initial time is at $q = 0$ with probability 1).

Translation symmetry. We have already assumed ρ independent of n and, thus, the random walk transition probability is invariant under time translation.

In addition, we now assume that the transition probability is also invariant under space translations and, thus,

$$\rho(q, q') \equiv \rho(q - q').$$

As a consequence, the evolution equation takes the form of the convolution equation,

$$P_{n+1}(q) = \int \mathrm{d}q' \, \rho(q - q') P_n(q'),$$

which also appears in the discussion of the central limit theorem of probabilities.

Local random walk. We consider only transition functions piecewise differentiable and with bounded variation, and satisfying a property of locality in the form of an exponential decay: we assume that the transition probabilities $\rho(q)$ satisfy a bound of exponential form,

$$\rho(q) \le M \, \mathrm{e}^{-A|q|}, \quad M, A > 0 \, .$$

Qualitatively, large displacements have a very small probability.

Fourier representation
The evolution equation simplifies after Fourier transformation. We thus introduce

$$\tilde{P}_n(k) = \int \mathrm{d}q \; \mathrm{e}^{-ikq} \, P_n(q) \, ,$$

which is a generating function of the moments of the distribution $P_n(q)$. The reality of $P_n(q)$ and the normalization of the total probability imply

$$\tilde{P}_n^*(k) = \tilde{P}_n(-k), \quad \tilde{P}_n(k=0) = 1 \, .$$

Similarly, we introduce

$$\tilde{\rho}(k) = \int \mathrm{d}q \; \mathrm{e}^{-ikq} \, \rho(q),$$

which is also a generating function of the moments of the distribution $\rho(q)$:

$$\langle q^r \rangle = \int \mathrm{d}q \, \rho(q) q^r .$$

Finally, the exponential decay condition implies that the function $\tilde{\rho}(k)$ is analytic in the strip $|\operatorname{Im} k| < A$ and, thus, has a convergent expansion at $k = 0$.

The evolution equation then becomes

$$\tilde{P}_{n+1}(k) = \tilde{\rho}(k) \tilde{P}_n(k)$$

and since with our choice of initial conditions $\tilde{P}_0(k) = 1$,

$$\tilde{P}_n(k) = \tilde{\rho}^n(k).$$

Generating function of cumulants

We introduce the generating function of the cumulants of $\rho(q)$,

$$w(k) = \ln \tilde{\rho}(k) \Rightarrow \; w^*(k) = w(-k), \quad w(0) = 0\,. \tag{2}$$

Then,

$$\tilde{P}_n(k) = \mathrm{e}^{nw(k)}, \quad P_n(q) = \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq} \, \tilde{P}_n(k) = \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq + nw(k)}\,.$$

The regularity of $\tilde{\rho}(k)$ and the condition $\tilde{\rho}(0) = 1$ imply that $w(k)$ has a regular expansion at $k = 0$ of the form

$$w(k) = -iw_1 k - \frac{1}{2} w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r\,,$$

where w_r is the rth cumulant, for example,

$$w_1 = \langle q \rangle\,, \quad w_2 = \left\langle q^2 \right\rangle - \langle q \rangle^2 = \left\langle (q - \langle q \rangle)^2 \right\rangle \geq 0\,.$$

Random walk: Asymptotic behaviour from a direct calculation

With the hypotheses satisfied by P_0 and ρ, the determination of the asymptotic behaviour for $n \to \infty$ follows from arguments identical to those leading to the central limit theorem of probabilities.

For $n \to \infty$ and $w_1 \neq 0$, $w(k)$ is dominated by the first term and, thus

$$P_n(q) \underset{n\to\infty}{\sim} \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - inw_1 k} = \delta(q - nw_1).$$

The random variable q/n converges with probability 1 toward its expectation value w_1 (the mean velocity).

For $n \to \infty$ and $w_1 = 0$, $w(k)$ is dominated by the term of order k^2 and thus

$$P_n(q) \underset{n\to\infty}{\sim} \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - nw_2 k^2/2} = \frac{1}{\sqrt{2\pi n w_2}} \mathrm{e}^{-q^2/2nw_2} \; .$$

Then it is the random variable $q/\sqrt{n}$ that has a limiting distribution, a Gaussian distribution with width $\sqrt{w_2}$.

The random variable that characterizes the deviation with respect to the mean trajectory,

$$X = (s - w_1)\sqrt{n} = \frac{q}{\sqrt{n}} - w_1\sqrt{n}\,, \tag{3}$$

and thus $\langle X \rangle = 0$, has, as limiting distribution, the universal Gaussian distribution

$$L_n(X) = \sqrt{n}P_n(nw_1 + X\sqrt{n}) \sim \frac{1}{\sqrt{2\pi w_2}}\,\mathrm{e}^{-X^2/2w_2},$$

which depends only on the parameter w_2.

The neglected terms are of two types, multiplicative corrections of order $1/\sqrt{n}$ and additive corrections decreasing exponentially with n.

The result implies that the mean deviation from the mean trajectory increases as the square root of time, a characteristic property of the Brownian motion.

Continuum time limit

The asymptotic Gaussian distribution of the deviation $\bar{q} = q - nw_1$ from the mean trajectory is

$$P_n(\bar{q}) \sim \frac{1}{\sqrt{2\pi n w_2}} \, \mathrm{e}^{-\bar{q}^2/2nw_2} \, .$$

By changing the time scale and by a continuous interpolation, one can define a diffusion process or Brownian motion in continuous time.

Let t and ε be two real positive numbers and n the integer part of t/ε:

$$n = [t/\varepsilon] \, . \tag{4}$$

One then takes the limit $\varepsilon \to 0$ at t fixed and thus $n \to \infty$.

If the time t is measured with a finite precision Δt, as soon as $\Delta t \gg \varepsilon$, time can be considered as a continuous variable for what concerns all expectation values of continuous functions of time.

One then performs the change of distance scale

$$\bar{q} = x/\sqrt{\varepsilon}.$$

Since the Gaussian function is continuous, the limiting distribution takes the form

$$P_n(q)/\sqrt{\varepsilon} \underset{\varepsilon \to 0}{\sim} \Pi(t,x) = \frac{1}{\sqrt{2\pi t w_2}}\, \mathrm{e}^{-x^2/2tw_2}\,. \tag{5}$$

(The change of variables $q \mapsto x$ implies a change of normalization of the distribution.) This distribution is a solution of a diffusion or heat equation:

$$\frac{\partial}{\partial t}\Pi(t,\mathbf{x}) = \tfrac{1}{2} w_2 \frac{\partial^2}{(\partial x)^2}\Pi(t,x).$$

In the limit $n \to \infty$ and in suitable macroscopic variables, one thus obtains a diffusion process that can entirely be described in continuum time and space.

The limiting distribution $\Pi(t, x)$ implies a scaling property characteristic of the Brownian motion. The moments of the distribution satisfy

$$\langle x^{2m} \rangle = \int \mathrm{d}x \, x^{2m} \Pi(t, x) \propto t^m. \tag{6}$$

The variable $x/\sqrt{t}$ has time-independent moments.

Dimensions. As the change $\bar{q} = x/\sqrt{\varepsilon}$ also indicates, one can thus assign to the position x a dimension $1/2$ in time unit (this also corresponds to assign a Hausdorff dimension two to a Brownian trajectory in higher dimensions).

Corrections to continuum limit

One can also study how perturbations to the limiting Gaussian distribution decrease with ε.

We can express the distribution of q in terms of $w(k) = \ln \tilde{\rho}(k)$. Correspondingly, we introduce $\bar{w}(k)$ the equivalent quantity for $\bar{q} = q - nw_1$. We then obtain the relation

$$\int \mathrm{d}k \ \mathrm{e}^{ik\bar{q}+iknw_1} \, \mathrm{e}^{nw(k)} = \int \mathrm{d}k \ \mathrm{e}^{ik\bar{q}+n\bar{w}(k)}$$

with

$$\bar{w}(k) = w(k) + ikw_1 \, .$$

With our assumptions, the expansion of the regular function $\bar{w}(k)$ in powers of k reads

$$\bar{w}(k) = -\frac{1}{2} w_2 k^2 + \sum_{r=3} \frac{(-i)^r}{r!} w_r k^r \, .$$

After the introduction of macroscopic variables, which for the Fourier variables corresponds to $k = \kappa\sqrt{\varepsilon}$, one finds

$$n\bar{w}(k) = t\omega(\kappa) \text{ with } \omega(\kappa) = -\frac{w_2}{2!}\kappa^2 + \sum_{r=3} \varepsilon^{r/2-1}\frac{(-i)^r}{r!} w_r \kappa^r .$$

One observes that, when $\varepsilon = t/n$ goes to zero, each additional power of κ goes with an additional power of $\sqrt{\varepsilon}$.

In the continuum limit, the distribution becomes

$$\Pi(t, x) = \frac{1}{2\pi}\int \mathrm{d}\kappa\, \mathrm{e}^{-i\kappa x}\, \mathrm{e}^{tw(\kappa)} .$$

Differentiating with respect to the time t, one obtains

$$\frac{\partial}{\partial t}\Pi(t, x) = \frac{1}{2\pi}\int \mathrm{d}\kappa\, w(\kappa)\, \mathrm{e}^{-i\kappa x}\, \mathrm{e}^{tw(\kappa)}$$

and in $w(\kappa)$, κ can then be replaced by the differential operator $i\partial/\partial x$.

One thus finds that $\Pi(t,x)$ satisfies the linear generalized partial differential equation

$$\frac{\partial}{\partial t}\Pi(t,x) = \left[\frac{w_2}{2!}\left(\frac{\partial}{\partial x}\right)^2 + \sum_{r=3}\varepsilon^{r/2-1}\frac{1}{r!}w_r\left(\frac{\partial}{\partial x}\right)^r\right]\Pi(t,x).$$

In the expansion, each additional derivative implies an additional factor $\sqrt{\varepsilon}$ and, thus, the contributions that contain more derivatives decrease faster to zero.

Random walk: Universality and fixed points of transformations

We now derive the universal properties of the asymptotic random walk, that is, the existence of a limiting Gaussian distribution independent of the initial distribution, and its scaling behaviour by a quite different method, which does not involve calculating the asymptotic distribution explicitly.

For simplicity, we assume that the initial number of time steps is of the form $n = 2^m$. The idea then is to recursively combine the time steps two by two, decreasing the number of steps by a factor two at each iteration. We then look for fixed points of such a transformation.

This method provides a simple application of RG ideas to the derivation of universal properties.

It also allows us to introduce some basic RG terminology.

Time scale transformation and renormalization

At each iteration one replaces $\rho(q-q')$ by

$$[\mathcal{T}\rho](q-q') \equiv \int \mathrm{d}q''\rho(q-q'')\rho(q''-q') = \int \mathrm{d}q''\rho(q-q'-q'')\rho(q''),$$

rescaling the time scale by a factor two.

The transformation of the distribution $\rho(q)$ is non-linear but applied to the function $w(k)=\ln\tilde{\rho}(k)$, it becomes the linear transformation since

$$[\mathcal{T}\tilde{\rho}](k)=\tilde{\rho}^2(k) \;\Rightarrow\; [\mathcal{T}w](k)\equiv 2w(k).$$

This transformation has an important property: it is independent of m or n. In the language of dynamical systems, its repeated application generates a stationary, or invariant under time translation, Markovian dynamics.

Large time behaviour and fixed points. The large time behaviour is obtained by iterating the transformation, studying $\mathcal{T}^m$ for $m \to \infty$.

A limiting distribution necessarily is a fixed point of the transformation.

It corresponds to a function $w_*(k)$ (the notation $_*$ is not related to complex conjugation) that satisfies

$$[\mathcal{T} w_*](k) \equiv 2w_*(k) = w_*(k).$$

For the class of fast decreasing distributions, the function $w_*(k)$ has an expansion in powers of k of the form ($w_*(0) = 0$)

$$w_*(k) = -iw_1 k - \tfrac{1}{2} w_2 k^2 + \sum_{\ell=3} \frac{(-i)^\ell}{\ell!} w_\ell k^\ell \,, \ w_2 \geq 0$$

and one verifies that such a transformation has, with our assumptions, only the trivial fixed point $w_*(k) \equiv 0$.

To the time rescaling must be associated a rescaling (a renormalization) of the random space variable q.

Random variable renormalization. Non-trivial fixed points can be reached if the transformation is combined with a renormalization of the distance scale, $q \mapsto zq$, with $z > 0$. We thus consider the transformation

$$[\mathcal{T}_z w](k) \equiv 2w(k/z)\,.$$

The transformation $\mathcal{T}_z$ provides a simple example of a RG transformation, a concept that we describe thoroughly in the framework of phase transitions.

The fixed point equation then becomes

$$[\mathcal{T}_z w_*](k) \equiv 2w_*(k/z) = w_*(k),$$

which determines the possible values of z and the corresponding functions $w_*(k)$.

Dimension of the random variable. Comparing the rescaling of time and the random variable q, one can attach to q a dimension d_q in time unit defined by

$$d_q = \ln z / \ln 2\,. \tag{7}$$

Fixed points: generic situation: $w_1 \neq 0$

The functions $w_*(k)$ has an expansion in powers of k of the form $(w_*(0) = 0)$

$$w_*(k) = -iw_1 k - \tfrac{1}{2} w_2 k^2 + \sum_{\ell=3} \frac{(-i)^\ell}{\ell!} w_\ell k^\ell \,, \ w_2 \geq 0 \,.$$

The generic situation corresponds to $w_1 \neq 0$. Expanding the RG equation, at order k one finds

$$2w_1/z = w_1 \ \Rightarrow \ z = 2 \,.$$

Then, identifying the terms of higher degree, one obtains

$$2^{1-\ell} w_\ell = w_\ell \ \Rightarrow \ w_\ell = 0 \text{ for } \ell > 1 \,.$$

Therefore, a fixed point solution is

$$w_*(k) = -iw_1 k \,.$$

The fixed points form a one-parameter family, but the parameter w_1 can also be absorbed into a normalization of the random variable q.

Since

$$\rho_*(q) = \frac{1}{2\pi}\int \mathrm{d}k\ \mathrm{e}^{ikq-iw_1k} = \delta(q-w_1),$$

fixed points correspond to the certain distribution $q = \langle q\rangle = w_1$.

Since space and time are rescaled by the same factor 2, q has dimension $d_q = \ln z/\ln 2 = 1$ in time unit.

Consistently, the fixed point corresponds to $q(t) = w_1 t$, the equation of the mean path.

Centred distribution

For a centred distribution, $w_1 = 0$ and one has to expand to order k^2. One finds the equation

$$w_2 = 2w_2/z^2.$$

Since the variance w_2 is strictly positive, except for a certain distribution, a case that we now exclude, the equation implies $z = \sqrt{2}$.

Again, the coefficients w_ℓ vanish for $\ell > 2$ and the fixed points have the form

$$w_*(k) = -\tfrac{1}{2} w_2 k^2 .$$

Therefore, one finds the Gaussian distribution

$$\rho_*(q) = \frac{1}{2\pi} \int \mathrm{d}k \; \mathrm{e}^{ikq - w_2 k^2/2} = \frac{1}{\sqrt{2\pi w_2}} \, \mathrm{e}^{-q^2/2w_2} .$$

The dimension $d_q = \ln z / \ln 2 = \frac{1}{2}$ is consistent with the scaling property $x \propto \sqrt{t}$ of the Brownian motion.

The two essential asymptotic properties of the random walk, convergence toward a Gaussian distribution and scaling property are thus reproduced by the RG type analysis.

Local and global stability of fixed points

For a non-linear RG transformation, a global stability analysis is in general impossible. One can only study the local stability of fixed points.

Here, since the transformation is linear local and global stabilities are equivalent. Setting

$$w(k) = w_*(k) + \delta w(k),$$

one finds,

$$[\mathcal{T}_z \delta w](k) \equiv 2\delta w(k/z)\,.$$

One then looks for the eigenvectors and eigenvalues of the transformation $\mathcal{T}_z$:

$$[\mathcal{T}_z \delta w](k) = \tau \delta w(k).$$

To the eigenvalue τ, one associates the exponent

$$\alpha = \ln \tau / \ln 2\,.$$

The perturbation δw has an expansion in powers of k of the form,

$$\delta w(k) = \sum_{\ell=1} \frac{(-i)^\ell}{\ell!} \delta w_\ell k^\ell \,.$$

Then,

$$[\mathcal{T}_z \delta w](k) = 2\delta w(k/z) = 2\sum_{\ell=1} \frac{(-ik)^\ell}{\ell!} z^{-\ell} \delta w_\ell \,.$$

The expression shows that the functions k^ℓ with $\ell > 0$ are the eigenvectors of the transformation $\mathcal{T}_z$ and the corresponding eigenvalues are

$$\tau_\ell = 2z^{-\ell} \;\Rightarrow\; \alpha_\ell = \ln \tau_\ell / \ln 2 = 1 - \ell \ln z / \ln 2 \,.$$

For $n = 2^m$ time steps, after m iterations, the component δw_ℓ is multiplied by n^{α_ℓ} since

$$\mathcal{T}_z^m k^\ell = \tau_\ell^m k^\ell = 2^{m\alpha_\ell} = n^{\alpha_\ell} k^\ell.$$

The behaviour, for $n \to \infty$, of the component of δw_ℓ on the eigenvector k^ℓ thus depends on the sign of the exponent α_ℓ for the various values of ℓ.

Fixed point stability: $w_1 \neq 0$

We now introduce the RG terminology to discuss eigenvectors and eigenvalues or corresponding exponents.

(i) $\ell = 1 \Rightarrow \tau_1 = 1$, $\alpha_1 = 0$. If one adds a term δw proportional to the eigenvector k to $w_*(k)$, $\delta w(k) = -i\delta w_1 k$, then

$$w_1 \mapsto w_1 + \delta w_1,$$

which correspond to a new fixed point. This change has also the interpretation of a linear transformation on k or on the random variable q.

An eigen-perturbation corresponding to the eigenvalue $\tau = 1$ and, thus to an exponent $\alpha = 0$, is called marginal.

(ii) $\ell > 1 \Rightarrow \tau_\ell = 2^{1-\ell} < 1$, $\alpha_\ell < 0$. The components of δw on such eigenvectors converge to zero for n or $m \to \infty$.

In the RG terminology, the eigen-pertubations that correspond to eigenvalues smaller in modulus than 1 and, thus, to negative exponents, are called irrelevant.

The notion of universality. Universality, in the RG formulation, is a consequence of the property that all eigenvectors, but a finite number, are irrelevant.

Line of fixed points. Quite generally, the existence of a one-parameter family of fixed points implies the existence of an eigenvalue $\tau = 1$ and, thus, an exponent $\alpha = 0$. Indeed, let us assume the existence of one-parameter family of fixed points $w_*(s)$,

$$\mathcal{T} w_*(s) = w_*(s)\,,$$

where $w_*(s)$ is a differentiable function of the parameter s. Then,

$$\mathcal{T}\,\frac{\partial w_*}{\partial s} = \frac{\partial w_*}{\partial s}.$$

Fixed point stability: $w_1 = 0$.
We now study the stability of the fixed point corresponding to the transformation $\mathcal{T}_{\sqrt{2}}$. One sets

$$w(k) = w_*(k) + \delta w(k),$$

and looks for the eigenvectors and eigenvalues of the transformation

$$[\mathcal{T}_{\sqrt{2}}\,\delta w](k) \equiv 2\delta w(k/\sqrt{2}) = \tau \delta w(k).$$

The eigenvalues are

$$\tau_\ell = 2^{1-\ell/2}.$$

The correspondent exponents are

$$\alpha_\ell = \ln \tau_\ell / \ln 2 = 1 - \ell/2\,.$$

The values can be classified as:

(i) $\ell = 1 \Rightarrow \tau_1 = \sqrt{2}$, $\alpha_1 = \frac{1}{2}$. This corresponds to an unstable direction; a component on such a eigenvector diverges for $m \to \infty$.

In the RG terminology, a perturbation corresponding to a positive exponent α, and which thus leads away from the fixed point, is called relevant.

Here, a perturbation linear in k violates the condition $w_1 = 0$. One is then brought back to the study of the more stable fixed points with $w_1 \neq 0$.

(ii) $\ell = 2 \Rightarrow \tau_2 = 1, \quad \alpha_2 = 0$. A vanishing eigenvalue characterizes a marginal perturbation. Here, the perturbation only modifies the value of w_2 and, again, has an interpretation as a linear transformation on the random variable.

(iii) $\ell > 2 \Rightarrow \tau_\ell = 2^{1-\ell/2} < 1\,, \quad \alpha_\ell = 1 - \ell/2 < 0$. Finally, all perturbations $\ell > 2$ correspond to stable directions in the sense that their amplitudes converge to zero for $m \to \infty$ and are irrelevant.

Redundant perturbations. In the examples examined here, the marginal perturbations correspond to simple changes in the normalization of the random variables. In many problems, this normalization plays no role. One can then consider that fixed points corresponding to different normalizations should not be distinguished.

From this viewpoint, in both cases one has found really only one fixed point. The perturbation corresponding to the vanishing eigenvalue is then no longer called marginal but redundant, in the sense that it changes only an arbitrary normalization.

Other universality classes. Other values of $z = 2^{1/\mu}$, correspond formally to new fixed points of the form $|k|^{\mu}$, $0 < \mu < 2$ ($\mu > 2$ is excluded because the coefficient of k^2 is strictly positive).

However, these fixed points are no longer regular functions of k. They correspond to distributions that have no second moment $\langle q^2 \rangle$ and thus no variance: they decay only algebraically for large values of q. In the RG terminology, they correspond to different universality classes, distributions with other decay properties.

Random walk on the lattice of points with integer coordinates. The analysis can also be generalized to a random walk on the points of integer coordinate. Then $w(k)$ is a periodic function of period 2π.

However, at each RG transformation the period is multiplied by a factor $z > 1$. Thus, asymptotically, the period diverges and, at least for continuous observables, the discrete character of the initial lattice disappears.

In the d-dimensional lattice $\mathbb{Z}^d$, if the random walk has hypercubic symmetry, the leading term in the expansion of $w(\mathbf{k})$ for $\mathbf{k}$ small is again $\frac{1}{2}w_2\mathbf{k}^2$ because it is the only quadratic hypercubic invariant. Therefore, asymptotically the random walk is a Brownian motion with rotation symmetry.

The lattice structure is only apparent in the first irrelevant perturbation because there exists two independent cubic invariant monomials of degree four:

$$\sum_{\mu=1}^{d} k_\mu^4, \quad \left(\mathbf{k}^2\right)^2.$$

Brownian motion and path integral

An iteration of the evolution equation of the translation invariant random walk

$$P_n(q) = \int \mathrm{d}q' \rho(q-q') P_{n-1}(q'),$$

in the case of a certain initial position $q = q_0 = 0$, yields

$$P_n(q) = \int \mathrm{d}q' \mathrm{d}q_1 \, \mathrm{d}q_2 \ldots \mathrm{d}q_{n-1} \, \rho(q-q_{n-1}) \ldots \rho(q_2 - q_1) \rho(q_1).$$

If one is interested only in the asymptotic properties of the distribution, which have been shown to be independent of the initial transition probability, one can obtain them, in the continuum limit, starting directly from Gaussian transition probabilities of the form (assuming rotation symmetry)

$$\rho(q) = \frac{1}{(2\pi w_2)^{1/2}} \, \mathrm{e}^{-q^2/2w_2} \, .$$

The iterated evolution equation becomes

$$P_n(q) = \frac{1}{(2\pi w_2)^{n/2}} \int \mathrm{d}q_1\, \mathrm{d}q_2\, \ldots \mathrm{d}q_{n-1}\, \mathrm{e}^{-\mathcal{S}(q_0,q_2,\ldots,q_n)} \tag{8}$$

with $q_n = q$ and

$$\mathcal{S}(q_0, q_2, \ldots, q_n) = \sum_{\ell=1}^{n} \frac{(q_\ell - q_{\ell-1})^2}{2w_2}.$$

We then introduce macroscopic time variables,

$$\tau_\ell = \ell\varepsilon\,, \quad \tau_n = n\varepsilon = t\,,$$

and a continuous, piecewise linear path $x(\tau)$ (Fig. 1)

$$x(\tau) = \sqrt{\varepsilon}\left[q_{\ell-1} + \frac{\tau - \tau_{\ell-1}}{\tau_\ell - \tau_{\ell-1}}\,(q_\ell - q_{\ell-1})\right] \quad \text{for } \tau_{\ell-1} \leq \tau \leq \tau_\ell\,.$$

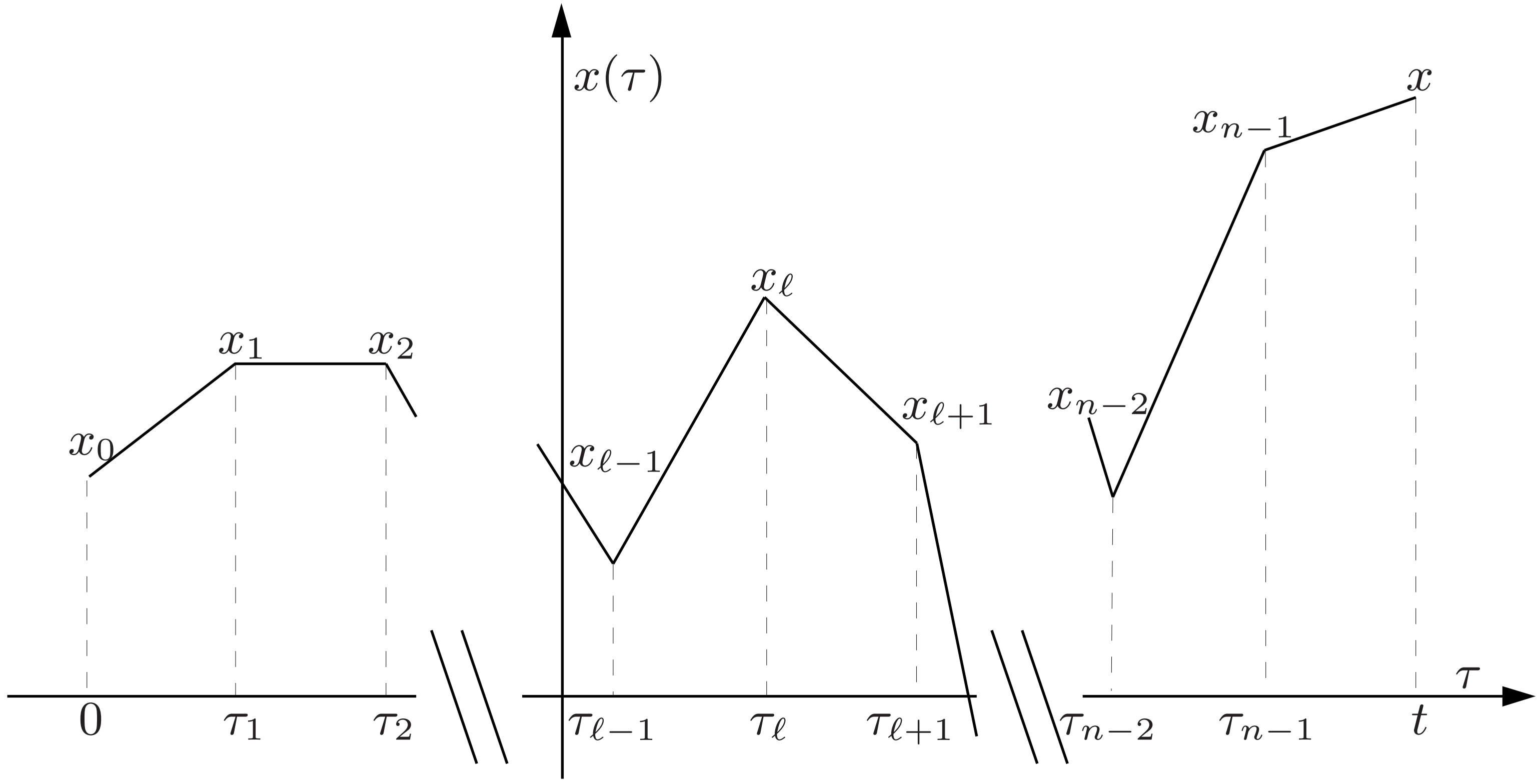

Fig. 1 Path contributing to the integral (8) $(d = 1)$ with $x_\ell \equiv x(\tau_\ell)$.

One verifies that $\mathcal{S}$ can be written as (with the notation $\dot{x}(\tau) \equiv \mathrm{d}x/\mathrm{d}\tau$)

$$\mathcal{S}\big(x(\tau)\big) = \frac{1}{2w_2}\int_0^t \big(\dot{x}(\tau)\big)^2 \mathrm{d}\tau$$

with the boundary conditions

$$x(0) = 0\,, \quad x(t) = \sqrt{\varepsilon}q = \mathbf{x}\,.$$

Moreover,

$$P_n(q) = \frac{1}{(2\pi w_2)^{1/2}} \int \left(\prod_{\ell=1}^{n-1} \frac{\mathrm{d}x(\tau_\ell)}{(2\pi w_2 \varepsilon)^{1/2}} \right) \mathrm{e}^{-\mathcal{S}(x)}\,.$$

In the continuum limit $\varepsilon \to 0$, $n \to \infty$ with t fixed, the expression becomes a representation of the distribution of the continuum limit,

$$\Pi(t, x) \sim \varepsilon^{-1/2} P_n(q),$$

in the form of a path integral, which we denote symbolically

$$\Pi(t, x) = \int [\mathrm{d}x(\tau)]\, \mathrm{e}^{-\mathcal{S}(x(\tau))}\,,$$

where $\int [\mathrm{d}x(\tau)]$ means sum over all continuous paths that start from the origin at time $\tau = 0$ and reach x at time t. The trajectories that contribute to the path integral correspond to a Brownian motion, a random walk in continuum time and space. The representation of the Brownian motion by path integrals, initially introduced by Wiener, is also called Wiener integral.

Exercises

Exercise 1

One considers a Markovian random walk on a two-dimensional square lattice. At each time step, the walker either remains motionless with probability $1 - s$, or moves by one lattice spacing in one of the four possible directions with the same probability $s/4$, where $0 < s < 1$. At initial time $n = 0$ the walker is at the point $\mathbf{q} = 0$.

Determine the asymptotic distribution of the walker position after n steps and calculate the asymptotic distribution for $n \to \infty$. What can be said about the space symmetry of the asymptotic distribution?

Exercise 2

One considers a Markovian random walk on a cubic lattice, that is, in $\mathbb{Z}^3$. At each step the walker either remains motionless with probability $1 - s$, or moves by one lattice spacing in one of the six possible directions with the same probability $s/6$, where $0 < s < 1$. At initial time $n = 0$ the walker is at the point $\mathbf{q} = 0$.

Determine the asymptotic distribution of the position of the walker after n steps when $n \to \infty$. What can be said about the space symmetry of the asymptotic distribution?

Exercise 3

A non-translation invariant evolution equation. One considers the evolution equation

$$P_n(q) = \int \mathrm{d}q' \rho(q - \sigma q') P_{n-1}(q'), \quad \sigma > 0\,.$$

Solve the problem explicitly in the large time limit.

Set up a RG formalism obtained by replacing two time steps by one. As a function of the parameter σ, determine the fixed point and the fixed point properties.

Some hints. To have an idea of the RG scheme, we iterate a first time:

$$\mathcal{T} = \int \mathrm{d}q'' \rho(q - \sigma q'')\rho(q'' - \sigma q')$$
$$= \int \mathrm{d}q'' \rho(q - \sigma^2 q' - \sigma q'')\rho(q'').$$

We then define the first transition function as $\rho_0(q)$ and the first σ parameter as σ_0. As boundary condition we have $\rho_0(q - \sigma_0 q')$.

We define

$$\rho_1(q) = \int \mathrm{d}q' \rho(q - \sigma_0 q')\rho(q')$$

and

$$\sigma_1 = \sigma_0^2$$

such that

$$\rho_0(q - \sigma_0 q') \mapsto \rho_1(q - \sigma_1 q').$$

Iterate.

Exercise 4

An example: local stability.

Consider the example

$$\rho(q) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-q^2/2} \, .$$

Starting from the RG equation

$$[\mathcal{T}\rho](q,q'') = \int \mathrm{d}q' \rho(q - \sigma q'')\rho(q'' - \sigma q'), \tag{9}$$

determine the value of the renormalization factor z for which the Gaussian probability distribution $\rho(q)$ is a fixed point of $\mathcal{T}_z$.

Substituting $\rho \mapsto \rho + \delta\rho$, expand equation (9) to first order in $\delta\rho$ and study the local stability of the fixed point. Show that the eigenvectors of the linear operator acting on $\delta\rho$ have the form

$$\delta\rho_p(q) = (\mathrm{d}/\mathrm{d}q)^p \, \rho(q), \quad p > 0 \, .$$

Infer the corresponding eigenvalues.

Exercise 5

Random walk on a circle. To exhibit the somewhat different asymptotic properties of a random walk on compact manifolds, it is proposed to study random walk on a circle. One still assumes translation invariance. The random walk is then specified by a transition function $\rho(q-q')$, where q and q' are two angles corresponding to positions on the circle. Moreover, the function $\rho(q)$ is assumed to be periodic and continuous. Determine the asymptotic distribution of the walker position. At initial time $n=0$, the walker is at the point $q=0$.

Exercise 6

Another universality class

One considers now the transition probability $\rho(q-q')$ with

$$\rho(q) = \frac{2}{3\pi}\frac{2+q^2}{(1+q^2)^2}.$$

The initial distribution is again

$$P_0(q) = \delta(q).$$

Evaluate the asymptotic distribution $P_n(q)$ for $n \to \infty$.