Analogue of cavity quantum electrodynamics for coupling between spin and a nanomechanical resonator: Dynamic squeezing and coherent manipulations

Fei Xue,1,* Ling Zhong,1 Yong Li,2,1 and C. P. Sun1,†
1Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100080, China
2Department of Physics and Astronomy, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland

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In the cavity QED analogue for the coupling system of a spin and a nanomechanical resonator, we show that, when the nanomechanical resonator is quantized, a spin-boson model for this coupling system can refer to a Jaynes-Cummings (JC) or an anti-JC model. These observations predict some quantum optical phenomena, such as dynamic squeezing in the single oscillation mode of the nanomechanical resonator. By modulating the phase of rf magnetic field, one can switch the system between the JC and anti-JC model for the detection of the single spin.

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INTRODUCTION

Many recent experiments have exhibited the possibilities to reach GHz mechanical oscillations by a nanomechanical resonator (NAMR).1–4 With such a high oscillating frequency at mK temperature, the NAMR can be modeled as a quantized harmonic oscillator. Then it can be used to explore the new phenomena in the quantum-classical crossover, to serve as a mechanical analogue of quantum optical devices,5,6 and as a quantum probe that can be viewed as a resonator: Dynamic squeezing and coherent manipulations quantum version of magnetic-resonance force microscopy (MRFM) for single electron spin detection.7,8

For conventional MRFM when the number of spins in the sample tends to approach 1 and the NAMR may reach quantum regime, a fully quantum model is expected to describe this coupling system correctly. In this paper, to treat the coherent coupling between NAMR and spin in a fully quantum way, we make use of an analogue of the Jaynes-Cummings (JC) model in cavity QED. Such an artificially engineered cavity QED system is used to demonstrate the feature of the quantized NAMR through the physical effects similar to various quantum optical phenomena. For example, in the large detuning limit that the NAMR does not exactly resonate with the spin, the virtual transition of the spin will result in the interesting squeezing effect for the NAMR. In the resonant case, the quantum dynamics of the coupling system is described by the JC or anti-JC model and thus a typical collapse-revival phenomenon can be exhibited by the mode of the NAMR.

MODEL SETUP

As illustrated in Fig. 1, a ferromagnetic particle is glued on the middle of the NAMR and exerts a gradient magnetic field on the spin in a static magnetic field \( B_0 \) pointing in the \( z \) direction. The spin is also exposed to a rotating rf magnetic field \( \vec{B}_r(t) = B_{rz}(\cos(\omega_r t + \phi), B_{r\perp} \sin(\omega_r t + \phi), 0) \) in the \( x-y \) plane. The magnetic tip produces a dipolar magnetic field \( \vec{B}_{\text{tip}} = \mu_0 [3(\vec{n} \cdot \vec{m}) \vec{n} - \vec{m})]/(4 \pi r^3) \) at the position of the spin. The magnetic moment of the ferromagnetic particle pointing in the \( z \) direction, and \( r \) is the distance between the tip and the spin. The coordinate of the spin is set as \((0, 0, d)\). For a small vibration of tip, the magnetic field at the position of the spin produced by the magnetic tip is approximately \( B_{\text{tip}}(z) = (A - Gz) \) along the \( z \) direction for the small deviation \( z \) of the tip. Here, \( A = \mu_0 m/(2\pi d^2) \), \( G = 3\mu_0 m/(2\pi d^3) \), and \( m_z = |\vec{m}| \).

The NAMR with the magnetic tip is modeled as a harmonic oscillator with effective mass \( m_{\text{eff}} \) and frequency \( \omega_c \). We introduce creation and annihilation operators \( a \) and \( a^\dagger \) by \( z = \lambda (a^\dagger + a) \), where \( \lambda = \sqrt{\hbar/(2m_{\text{eff}} \omega_c)} \). In the rotating reference frame where the time dependence due to \( \vec{B}_r(t) \) is eliminated, the Hamiltonian of the coupling system can finally be written as

\[
H = \hbar \omega_c a^\dagger a + \gamma h B_1(z) (\cos \varphi S_z + \sin \varphi S_y) + \hbar g (a^\dagger S_z + a S_z + a^\dagger S_z + a S_z), \tag{1}
\]

where \( \gamma \) is the gyromagnetic ratio and \( g = \gamma A G / 2 \). \( S_z = S_\pm i S_y \) are defined by Pauli matrices \((1/2)\sigma_x \rightarrow S_z, (1/2)\sigma_y \rightarrow S_y, \) and \((1/2)\sigma_z \rightarrow S_z \) for the spin. We have also set \( B_0 + A + \omega_c / \gamma = 0 \) with special parameters so that the model can be simplified. This is a typical spin-boson model, an analogue of the JC model in cavity QED. It can be used to

FIG. 1. (Color online) The setup of the device. A spin and an NAMR are coupled with each other by a tiny ferromagnetic particle attached to the NAMR.
demonstrate rich quantum optical phenomena on-resonance and off-resonance.

**DYNAMIC SQUEEZING**

We first consider the case of the off-resonance situation in which the squeezing effects of oscillations of the NAMR appear because of the interference between the two oscillation modes with respect to the two eigenstates of the spin. This may reduces the uncertainty of position or momentum below the standard quantum limit. A similar phenomenon has been investigated for a NAMR that capacitively couples to its substrate, and an NAMR that couples to a Cooper pair box.

We propose \( \omega_s < \omega_c = \gamma B_1 \), as \( \omega_c \) is about 10 MHz with a quality factor of 10^4 in some experiments. In the following discussion we have put \( \varphi = 0 \) and \( \hbar = 1 \) to display squeezing effects. In large detuning limit \( g < \Delta = |\omega_s - \omega_c| \), the Hamiltonian can be approximately written in a diagonal form \( \hat{H}_{\text{diag}} = \hat{H}_1 \otimes |1\rangle\langle 1| + \hat{H}_0 \otimes |0\rangle\langle 0| \) by adiabatically eliminating coherent effect between the spin states \(|0\rangle \) and \(|1\rangle \). \( \hat{H}_k \), \( k = 0,1 \) reads

\[
\hat{H}_k = \omega_c a_k^\dagger a_k - g_k (a_k^\dagger a_k + 2a_k a_k^\dagger),
\]

where \( g_k = (-1)^k g^2 / (4\Delta) \) and the constant terms \((-1)^k \omega_c / 2 - g_k \) have been omitted.

Diagonalization of \( \hat{H}_k \) gives \( \hat{H}_k = \Omega_k b_k^\dagger b_k + \epsilon_k \). The Bogoliubov excitation by \( b_k = \mu_k a_k - \nu_k a_k^\dagger \) is defined by squeezing transformation with \( \mu_k = (\sqrt{N_k} + i/\sqrt{N_k})/2, \nu_k = (\sqrt{N_k} - 1/\sqrt{N_k})/2, N_k = (\omega_c / \omega_s - (-1)^k g^2 / 2\omega_s \Delta) \). The eigenfrequency \( \Omega_k = \omega_c [1 - (-1)^k g^2 / (2\omega_s \Delta)] \) contains dispersive shift \( \delta = g^2 / (2\omega_s \Delta) \) due to the Lamb effect, while the corresponding excitation is shifted by \( \epsilon_k = (-1)^k [\omega_c / 2 - g^2 / (4\Delta)] \).

Starting from a quasiclassical state, the coherent state \( |\alpha\rangle \) of mode \( a \), at time \( t \) the NAMR will evolve into the partial wave function \( \sqrt{\Psi(t)} = \exp(-i\Omega_k b_k^\dagger b_k) |\alpha\rangle \) with respect to each spin state \(|\pm\rangle\). To show the dynamic squeezing, \( \sqrt{\Psi(t)} \) is explicitly calculated by considering the fact \( B_k(t)|\Psi(t)|_k = \alpha |\Psi(t)|_k \) from the Heisenberg operators \( B_k(t) = \mu_k(t) b_k + \nu_k(t) b_k^\dagger \) for \( B_k(t) = a \), where \( \mu_k(t) = \mu_k \exp(i\Omega_k t) \) and \( \nu_k(t) = \nu_k \exp(i\Omega_k t) \). We further explicitly write quasiexcitation operator \( B_k(t) \) in terms of the elementary photon excitation

\[
B_k(t) = u_1(t) a_k + u_2(t) a_k^\dagger,
\]

where \( u_1(t) = \cos\Omega_k t + i(\mu_k^2 + \nu_k^2) \sin\Omega_k t \) and \( u_2(t) = 2i \mu_k \nu_k \sin\Omega_k t \). Thus in the case of off-resonance, the time evolution of the NAMR follows the eigenstate of the quasiexcitation operator \( B_k(t) \). And the eigenstate of \( B_k(t) \) is found to be a squeezed state with respect to the elementary excitation of the NAMR of \( a \) and \( a^\dagger \). The squeezing factor is calculated as

\[
r_k(t) = \ln(2\mu_k \nu_k \sin\Omega_k t + \sqrt{1 + 4\mu_k^2 \nu_k^2 \sin^2\Omega_k t}).
\]

The dynamic squeezings of oscillations in the NAMR are illustrated by plotting the curves of \( \gamma_k \) and \( \gamma_1 \) in Fig. 2(a). Depending on the spin state \(|k\rangle \), the NAMR is dynamically squeezed with different frequencies \( \Omega_k \) and different maximum squeezing factor \( \ln N_k \). The maximum squeezing factor of the NAMR oscillation depends on \( g / \Delta \) and the strength of coupling \( g / \omega_c \), which is shown in Fig. 2(b). In the plot we have assumed \( \omega_s < \omega_c \). The greater squeezing occurs with the smaller detuning and stronger coupling. To guarantee the large detuning assumption, the detuning cannot be too small. However, large squeezing can still be attained with optimal coupling strengths.

In large detuning situation, this spin-boson system also exhibits "collapse-revival" phenomenon. For the NAMR initially prepared in the coherent state \(|\alpha\rangle\), its position \( \zeta \) can be calculated according to the fact that the time evolution of NAMR always follows the eigenstate of \( B_k(t) \). Therefore we expand \( a + a^\dagger \) in terms of \( B_k(t) \) and \( B_k^\dagger(t) \), and then \( \langle \zeta \rangle \propto \langle a + a^\dagger \rangle \) is obtained as

\[
\langle \zeta \rangle \propto \sum_k 2c_k c_k^\dagger \Re \left( a \left[ \frac{1}{N_k} \cos\Omega_k t - i \sin\Omega_k t \right] \right).
\]

Here, the spin is initially prepared in the state \( c_0|0\rangle + c_1|1\rangle \). The oscillation of the NAMR collapses and revives, as illustrated in Fig. 2(c), is the result of the interference effect of the two oscillating mode \( \Omega_k \) labeled by the spin state \(|k\rangle\), which is a witness of the quantization of the NAMR.

**JC AND ANTI-JC PHASES**

Next we consider the situation that the spin resonates with the NAMR under the rotating-wave approximation. We show that, by modulating the phase \( \varphi \) of the rf magnetic field \( B_1(t) \), the device can be switched between the JC model and the anti-JC model. Such a structure offers a new type of mechanical analogue of cavity QED among various solid-state cavity QED systems.

In the following we assume \( B_1 > 0 \). When the phase of rf field \( \varphi = 0 \), the Hamiltonian (1) becomes of JC type,
in the rotating-wave approximation, where \( \omega_s = \gamma B_1 > 0 \) and the constant term \( h \omega / 2 \) is dropped. This can be understood in the interaction picture, in which the terms concerning \( a_S \) and \( a_S^\dagger \) with frequencies \( \pm (\omega_s + \omega_c) \) oscillates fast with higher frequency, and thus can be neglected. For this JC Hamiltonian in the large detuning limit \( g / \Delta \ll 1 \), where \( \Delta = |\omega_c - \omega_s| \), the dressed eigenfrequency of the NAMR is shifted as \( \omega + 2g^2(S)/\Delta \). The phase of the rf magnetic field \( \varphi = \pi \), the Hamiltonian \((1)\) becomes an anti-JC Hamiltonian

\[
H_{JC} = h \omega_s a^\dagger a + h \omega_c S_z + h g (a^\dagger S_+ + a S_-),
\]

in the rotating-wave approximation. Here \( \omega_s = -\gamma B_1 < 0 \). This is understood in the interaction picture, in which the terms concerning \( a_S \) and \( a_S^\dagger \) with frequencies \( \pm (\omega_s - \omega_c) \) rather than \( a_S \) and \( a_S^\dagger \), oscillates fast with higher frequency, thus should be neglected. For this anti-JC Hamiltonian at the large detuning situation, the dressed energy level of the NAMR reads \( \omega_s - 2g^2(S)/\Delta \).

In the two situations of the JC and anti-JC phases discussed above, the ac Stark shifts have opposite signs. This motivates us to present a scheme for the detection of single spin. The trick is similar to the interrupted oscillating the cantilever-driven adiabatic reversals (iOSCAR) protocol, and provides a convenient frequency shift detection without requiring the difficult measurement of the absolute frequency of the NAMR. The rf magnetic field is interrupted for half a cycle periodically as illustrated in Fig. 3(c). Then the phase of the rf magnetic field \( \varphi \) will swing between 0 and \( \pi \) correspondingly, which is illustrated in Fig. 3(b). Therefore the NAMR-spin system can switch between the JC and anti-JC models. When the switching is fast enough so that the spin only evolves little and is kept in its original state, i.e., the expectation of the spin \( \bar{S} \) keeps its value, the frequency shift of the NAMR swings between \( 2g^2(S)/\Delta \) and \( -2g^2(S)/\Delta \) periodically as the consequence of this Hamiltonian switching, which is illustrated in Fig. 3(a).

which is closed for the “quadratic” operators such as \( X = i(S^x a + S^x a^\dagger) \). Here the role of the bath is reflected by the average thermal photons \( n_{th} \) and the average \( \langle \cdot \cdot \cdot \rangle_E \) over the environment. The above close system of equations can be solved by the Laplace transform method numerically. In Fig. 4, for weak coupling to the bath, we plot the time evolution of the expectation of the energy of the NAMR \( \langle a^\dagger a \rangle \) with the initial condition \( n_{th} = 10, \langle a^\dagger a \rangle + \langle S_z \rangle = 11 \), \( X = \langle a^\dagger S a \rangle = 0 \). The case of stronger coupling to the environment, i.e., larger \( \kappa / g \), is also studied. It is not surprising that the oscillation of the energy of the NAMR dies out faster with larger \( \kappa / g \). With sufficient strong coupling to environment, no oscillation of the energy of the NAMR could be observed. This simple

\[
\frac{d}{dt} \langle a^\dagger a \rangle = -\kappa \langle a^\dagger a \rangle + gX + kn_{th},
\]

\[
\frac{d}{dt} \langle S_z \rangle = -2gX,
\]

\[
\frac{d}{dt} X = g \langle S_z \rangle - \frac{1}{2} \kappa X + 2g \langle a^\dagger S a \rangle + g
\]

\[
\frac{d}{dt} \langle a^\dagger S a \rangle = -gX - \kappa \langle a^\dagger S a \rangle
\]
consideration suggests that, even for nonzero temperature environment, it is still possible to observe the energy oscillation between the NAMR and the spin as long as the coupling to the environment is weak enough.

**EXPERIMENTAL FEASIBILITIES**

Finally let us address some issues in the experiments. The frequency shifts of the NAMR depends on the coupling constant \( g \sim 10^{-7} \) T/m, \( k_{eff} \) with \( \gamma = 10^{10} \) and \( k_{eff} \) the effective spring constant of NAMR.\(^8\) Using the experimental parameters, \( G \sim 10^5 \) T/m, \( k_{eff} = 0.11 \) mN/m, and \( \omega_s = 5.5 \) kHz, we obtain a frequency shift of about 7 Hz for the NAMR where \( g/\Delta = 0.1 \). The frequency shift is larger than that in the current OSCAR technique (about 4 mHz) by three orders of magnitude. Therefore when the temperature is reduced and the mechanical oscillation in the MRFM is quantized, the mechanical QED model provides an efficient protocol of single electron-spin detection. To reach such a scale of low temperature, there are some protocols\(^{25-27}\) that promise to cool the temperature of the NAMR down to mK or even lower. However, reducing the temperature of the NAMR still remains a great challenge in the experiments. If one can set the frequency of the NAMR at tens MHz, then tens mK temperature is sufficient to prepare the NAMR in the ground state. And the frequency shift to be detected is about tens Hz. The required resolving power is about \( 10^{-6} \), which is already achievable in the current experimental conditions.\(^{15}\) Therefore our scheme is realizable with optimal combination of current experimental techniques.

In summary, this paper studied a mechanical analog of cavity QED structure with a quantized NAMR. We theoretically predict some analogues of quantum optical phenomena, such as the dynamic squeezing. For practices in the experiments, by considering the influence of thermal decoherence, we suggest that coherent manipulation of JC and anti-JC phases might provide a protocol for the detection of single spin.

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