New $R$-matrices for the Yang–Baxter equation associated with the representations of the quantum superalgebra $U_{q,osp}(1, 2)$ with $q$ a root of unity

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Both the highest weight representations and the cyclic representation of the quantum superalgebra $U_{q,osp}(1, 2) = A$ are explicitly constructed in this Letter. By making use of the obtained finite dimensional representations with highest weights, the new $R$-matrices (solutions) of the Yang–Baxter equation are obtained through the universal $R$-matrix. In the main discussion of this paper emphasis is placed on the case that $q$ is a root of unity.

1. Introduction

Through the so-called universal $R$-matrix [1], the $R$-matrices (solutions) for the Yang–Baxter equation (YBE) can be constructed in terms of irreducible representations of both the quantum algebra [1] and the quantum superalgebra [2]. These $R$-matrices are called standard $R$-matrices if the representations used are irreducible and $q$ is not a root of unity. Recently, some new $R$-matrices, which possess new eigenvalues or new block diagonal structures, were obtained in terms of the non-generic finite dimensional representations of the case that $q$ is a root of unity [3]. Because they do not have a classical limit, these representations are “completely quantum”. However, these new $R$-matrices obtained are only associated with the quantum algebras rather than the quantum superalgebras.

In this Letter we try to construct the new matrices associated with the quantum superalgebra $A = U_{q,osp}(1, 2)$. To this end we systematically obtain its new finite dimensional representations and the cyclic representation thereby in the case that $q$ is a root of unity.

2. The algebraic relations for $A$

The quantum superalgebra $A$ is an associative algebra over the complex number field $C$ and generated by the elements $V_\pm$ and $H$ satisfying

$$V_+V_- + V_-V_+ = -\frac{1}{2}[2H], \quad [H, V_\pm] = \pm\frac{1}{2}V_\pm. \quad (1)$$

For convenience, we define $e_\pm = 2V_\pm$ and $\hat{h} = 2H$. It follows from eq. (1) that

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\[ e_+ e_-^{m+1} = (-1)^m e_- e_+^{m+1} + (-1)^{m+1} e_- e_+^m \frac{\hbar}{2} + m e_+^m \frac{\hbar}{2}, \]
\[ c_{2m}(\hbar) = \left( [m+1] - [m] \right) [\hbar - m], \quad c_{2m+1}(\hbar) = [m+1]([H-m] - [H-m-1]). \tag{2} \]

The above algebraic relations are proved by induction and are useful for the construction of representations for A.

3. New finite dimensional representations of A

Now we consider the Verma space \( V(\lambda) = \text{Span}\{ |m\rangle = e_+^m |\lambda\rangle |m\in \mathbb{Z}^+ = \{0, 1, 2, ...\} \}, \) where the highest weight vector \( |\lambda\rangle \) satisfies
\[ e_+ |\lambda\rangle = 0, \quad \hbar |\lambda\rangle = \lambda |\lambda\rangle, \quad \lambda \in \mathbb{C}. \]

Using eqs. (2) we obtain an infinite dimensional representation
\[ e_+ |2m\rangle = [m]([\lambda + 1 - m] - [\lambda - m]) |2m-1\rangle, \quad e_+ |2m+1\rangle = ([m] - [m+1]) [\lambda - m] |2m\rangle. \tag{3} \]
\[ e_- |2m\rangle = |2m-1\rangle, \quad \hbar |m\rangle = (\lambda - m) |m\rangle. \]

The above representation is called the Verma representation. For generic \( q \) (\( q^p \neq 1, p=1, 2, 3, ... \)), it is irreducible if \( \lambda \) is not a positive integer; it induces a finite dimensional representation on a certain quotient space corresponding to an invariant subspace in \( V(\lambda) \) if \( \lambda \in \mathbb{Z}^+ \). The latter has been well discussed in ref. [2].

Let us discuss the case that \( q \) is a root of unity, i.e., \( q^n = \pm 1 \) (\( n=2, 3, 4, 5, ... \)). Then, the fact that \( e_+ |2kp\rangle = 0 \) defines an invariant subspace \( V_{kp} = \text{Span}\{ |2kp+m\rangle |m\in \mathbb{Z}^+ \}. \) On the \( 2kp \)-dimensional quotient space \( Q(\lambda) = V(\lambda)/V_{kp} = \text{Span}\{ |j, m\rangle = |j+m\rangle |m=j, j-1, ..., -j\} \) for \( j= kp \frac{1}{2} \), the representation (3) induces a \( (2j+1) \)-dimensional representation \( \rho^{(k)}(\lambda) \):
\[ e_+ |j, m\rangle = |j+m\rangle ([\lambda + 1 - j - m] - [\lambda + 1 - j - m]) |j, m-1\rangle, \quad \text{for even } j+m, \]
\[ = [\lambda - j - m] ([j+m] - [j+m+1]) |j, m-1\rangle, \quad \text{for odd } j+m, \]
\[ e_- |j, m\rangle = |j, m+1\rangle, \quad \hbar |j, m\rangle = m |j, m\rangle. \tag{4} \]

It can be proved that the representation is indecomposable for \( k \geq 2 \) and irreducible for \( k = 1 \). However, this representation has no classical limit and is “completely quantum” because as \( q \to 1 \) eqs. (4) no longer define a representation for A.

4. The cyclic representation of A

Recently, the \( p \)-dimensional cyclic representation of the quantum algebra \( U_q sl(2) \) has been studied at \( q^p = 1 \).

The \( (J_\pm)^p \) of the generators \( J_\pm \), besides the Cartan element \( J_0 \), are non-zero constants [4] in the representation, that is to say, the \( (J_\pm)^p \) are no longer nilpotent.

Considering that \( (e_+)^{2p} \) and \( (e_-)^{2p} \) belong to the center of the subalgebra generated by \( e_+ \) and \( K^\pm = q \pm k \) at \( q^p = 1 \), we can extend the quotient space \( Q(\lambda)_{j=p+1/2} \) to obtain a “cyclic space” \( V_\lambda \) spanned by \( \{ |0\rangle, |1\rangle, |2\rangle, ..., |2p-1\rangle \}. \) On the space \( V_\lambda \) we obtain a cyclic representation
\[ e_+ |2m\rangle = [m]([\lambda + 1 - m] - [\lambda - m] + q|\eta|) |2m-1\rangle, \quad 0 \leq m \leq p-1, \]
\[ e_+ |2m+1\rangle = ([m] - [m+1]) ([\lambda - m] - q|\eta|) |2m\rangle, \quad 0 \leq m \leq p-2, \]
\[ e_+ |0\rangle = q^{\frac{p}{2}} |2p-1\rangle, \quad e_- |\tilde{m}\rangle = |\tilde{m}+1\rangle, \quad e_- |2p-1\rangle = |\eta| |0\rangle, \quad k \pm |m\rangle = q^{\pm k} |m\rangle. \tag{5} \]
It can be directly checked that eqs. (5) determine the representation matrices that indeed satisfy the basic relations (1). The special case with \( \zeta \) and \( \eta = 0 \) is just the irreducible representation defined by eq. (4) when \( k = 1 \).

5. \( R \)-matrices for \( A \)

According to Kulish, Reshetikhin and Saleur the quantum superalgebra \( A \) can be endowed with a Hopf algebraic structure by introducing an appropriate co-product, antipode and co-unit. From the quantum double theory the universal \( R \)-matrix is constructed in an explicit form,

\[
\mathcal{R} = q^{k\otimes k} \sum_{n=0}^{\infty} (q^{1/2} - q^{-3/2})^n (-1)^{n(n+1)/2} q^{-n(n-1)/4} \binom{[\frac{1}{2} n]}{+} q^{n(k-1)\otimes k} (e_-) \otimes (e_+)^n .
\]

where

\[
[x]_+ = \{ q^x - (1)^{2x} q^{-x} \} (q^{1/2} + q^{-1/2})^{-1}, \quad \frac{1}{2} [n]_+ = \frac{1}{2} [n] + \frac{1}{2} (n-1) + \frac{1}{2} (n-2) + \ldots \frac{1}{2} + \frac{1}{2} .
\]

Using representation (4), on the product space \( Q^\mu(\lambda) \otimes Q^\mu(\mu) \), we obtain the \( R \)-matrix

\[
R^{\mu\nu}(\lambda, \mu) = p^{\mu(\lambda)} \otimes p^{\mu(\mu)} (\mathcal{R}) .
\]

The explicit expression for the matrix elements of \( R^{\mu\nu}(\lambda, \mu) \) is defined as follows,

\[
R^{\mu\nu}(\lambda, \mu)(|j_1, m_1 \otimes j_2, m_2\rangle) = \sum_{m_1', m_2'} (R^{\mu\nu}(\lambda, \mu))^{m_1' m_2'}_{m_1 m_2} (|j_1, m_1' \otimes j_2, m_2'angle).
\]

\[
(R^{\mu\nu}(\lambda, \mu))^{m_1' M_2}_{m_1 M_2} = q^{(j_1 - m_1') (\mu - M_2 - j_2)} \left( \delta_{m_1}^{m_1'} \delta_{M_2}^{M_2'} + \sum_{n \geq 1} f(n) q^{-n(j_1 - m_1' + j_2 - M_2 - 2) + n (j_1 + j_2) + n (j_1 + j_2 + M_2 + 2) / 2} \delta_{m_1}^{m_1'} \delta_{M_2}^{M_2'} d(n)_{M_2'}^{M_2} \right) ,
\]

\[
M_2, M_2' \in \{ \ldots - j_2, j_2 \} ,
\]

where

\[
f(n) = 4^n (q^{1/2} - q^{-3/2})^n (-1)^{n(n+1)/2} q^{-n(n-1)/4} \binom{[\frac{1}{2} n]}{+} q^{-n(k-1)\otimes k} (e_-) \otimes (e_+)^n .
\]

\[
d(n)_{M_2}^{M_2'} = (-1)^{n/2} \prod_{l=1}^{n/2} [m_2 + j_2 - l + 1] \{ [\mu - j_2 - m_2 + l - 1] [\mu - j_2 - m_2 + l - 1] \}
\]

\[
\times ([j_2 + m_2 + 2 - l] [j_2 + m_2 + l + 1]) \delta_{M_2 - n/2}^{M_2'} ,
\]

\[
M_2' = 2m_2' + 1 + j, \quad M_2 = 2m_2 + 1 + j, \quad \text{for even } n ,
\]

\[
= (-1)^{(n-1)/2} \prod_{l=1}^{(n+1)/2} [m_2 + j_2 - l + 2] \{ [\mu - j_2 - m_2 + l + 1] [\mu - j_2 - m_2 + l - 2] \}
\]

\[
\times \prod_{k=1}^{(n-1)/2} [\mu - j_2 - m_2 + k - 1] ([j_2 + m_2 - k + 2] - [j_2 + m_2 - k + 1]) \delta_{M_2' - n/2}^{M_2} .
\]

\[
M_2' = 2m_2' + 1 + j, \quad M_2 = 2m_2 + 2 + j, \quad \text{for odd } n .
\]
\[ d(n)M_{ij}^n = (-1)^{n/2} \prod_{l=1}^{n/2} [\mu - j_l - m_l + l] \left[ (j_l + m_l - l + 1) - [j_l + m_l - l - 1] \right] \left( j_l + m_l - l + 1 \right) \]

\[ \times \left( [\mu - j_l - m_l + l] - [\mu - j_l - m_l + l - 1] \right) \delta_{m_{n/2}}^{j_{n/2}} , \]

\[ M_2 = 2m_2 + j_2, \quad M_2 = 2m_2 + j_2, \quad \text{for even } n , \]

\[ = (-1)^{(n + 1)/2} \prod_{k=1}^{(n + 1)/2} \left[ \mu - j_k - m_k + l - 1 \right] \left( [j_k + m_k - l + 2] - [j_k + m_k - l + 1] \right) \]

\[ \times \prod_{k=1}^{(n - 1)/2} \left[ m_k + j_k - l - 1 \right] \left( [\mu - j_k - m_k + l] - [\mu - j_k - m_k + l - 1] \right) \delta_{m_{(n - 1)/2}}^{j_{(n - 1)/2}} , \]

\[ M_2 = 2m_2 + j_2, \quad M_2 = 2m_2 + 1 + j_2, \quad \text{for odd } n . \]

For example, in the case that \( q^4 = 1 \) and \( j_1 = j_2 = \frac{1}{2} \), we introduce a continuous parameter \( t_q = q^2(\lambda - 1) \) and then have

\[ R(\lambda, \mu) = R^{3/2, 3/2}(\lambda, \mu) = q^{3\mu - 3(\lambda + \mu)} \text{ block diag}(A_1, A_2, A_3, A_4, A_3', A_2', A_1') , \]

where

\[ A_1 = -1, \quad A_1' = qt^{3/2}t_{\mu}^{3/2} , \]

\[ A_2 = \begin{pmatrix} -t_{\mu}^{1/2} & -t_{\mu}^{1/4}t_{\mu}^{-1/4} & -t_{\mu}^{1/2} \cr -t_{\mu}^{3/2}t_{\mu}^{3/4} & -t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \cr t_{\mu}^{1/2}t_{\mu}^{1/4} & t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \end{pmatrix} , \quad A_2' = \begin{pmatrix} t_{\mu}^{1/2}t_{\mu}^{3/4} & t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \cr t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \cr t_{\mu}^{1/2}t_{\mu}^{3/4} & t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \end{pmatrix} , \]

\[ A_3 = \begin{pmatrix} t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} \cr t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} \cr t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} \end{pmatrix} , \quad A_3' = \begin{pmatrix} -q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} \cr q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} \cr q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} \end{pmatrix} , \]

\[ A_4 = \begin{pmatrix} -t_{\mu}^{3/2} & t_{\mu}^{1/4}t_{\mu}^{-1/4} & t_{\mu}^{1/2}t_{\mu}^{3/4}t_{\mu}^{-1/4} \cr t_{\mu}^{1/2}t_{\mu}^{3/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} & t_{\mu}^{1/2}t_{\mu}^{3/4} \cr t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} & q^{-1/2}t_{\mu}^{1/4}t_{\mu}^{-1/4} \end{pmatrix} . \]

In fact, using the extended Kauffman diagram theory [5], we can verify that the YBE

\[ R_{12}(\lambda, \mu)R_{13}(\lambda, \nu)R_{23}(\mu, \nu) = R_{23}(\mu, \nu)R_{13}(\lambda, \nu)R_{12}(\lambda, \mu) \]

is indeed satisfied by the R-matrix (9). It is worthwhile to point out that in the YBE (10) the graded tensor product law \((a \otimes b) \cdot (c \otimes d) = (-1)^{p(x)p(y)}(ac \otimes bd)\) must be taken into account where \( p(x) = 0, 1 \) is the parity of a homogeneous element \( x \in A \).

We also notice that Jimbo obtained a similar R-matrix of \( U_q(sl(2)) \) for the quantum double of \( U_q(sl(2)) \) with two independent “Cartan like” elements. We hope the same construction can be made for the quantum superalgebra \( A \) [6].
6. Discussion

Finally, let us discuss some possible physical applications of the new \( R \)-matrices obtained in this Letter for the quantum superalgebra \( U_q\mathfrak{osp}(1,2) \).

(i) A recent work of Date et al. \[4\] shows that new solutions of the YBE for cyclic representations of quantum algebras (e.g. \( \mathfrak{sl}_n(n) \)) at \( q^p = 1 \) are associated with the generalized chiral Potts model. It is natural to expect that the supersymmetry version of their result can be obtained from the cyclic representations of the quantum superalgebra \( U_q\mathfrak{osp}(1,2) \) or the obtained new \( R \)-matrices.

(ii) Like the \( R \)-matrices for quantum algebras, our \( R \)-matrices for the quantum superalgebra can also be related to the Boltzmann weights of the integrable vertex model, but further work is needed to this end. In fact, if we take \( \lambda_1 \) and \( \lambda_2 \) in \( R_{ijkl}^{R\mathcal{H}_1}(\lambda_1, \lambda_2) \) as spectral parameters, the \( R \)-matrices \( R_{ijkl}^{R\mathcal{H}_1}(\lambda_1, \lambda_2) \) only include a discontinuous parameter \( q \). Thus, we need to construct the continuous parameter-dependent \( R \)-matrices \( R_{ijkl}^{R\mathcal{H}_2}(\lambda_1, \lambda_2; x) = R_{ijkl}^{R\mathcal{H}_2}(x) \) from \( R_{ijkl}^{R\mathcal{H}_1}(\lambda_1, \lambda_2) \), which satisfy the spectrum-dependent YBE

\[
R_{ijkl}^{R\mathcal{H}_2}(u)R_{ljnk}^{R\mathcal{H}_2}(v)R_{ijkl}^{R\mathcal{H}_2}(u-v)=R_{ijkl}^{R\mathcal{H}_2}(v)R_{ljnk}^{R\mathcal{H}_2}(u+v)R_{ijkl}^{R\mathcal{H}_2}(u) .
\]

The scheme of the Yang–Baxterization may be used for this purpose \[7\]. Then, decomposing \( R_{ijkl}^{R\mathcal{H}_2}(x) \) into a sum of projectors for eigenspaces of \( R_{ijkl}^{R\mathcal{H}_2}(x) \), \( R_{ijkl}^{R\mathcal{H}_2}(x)=\sum_k c_k(x) \hat{P}_k \), we can get the Hamiltonian \( H = \sum_{j<k} \hat{H}_{j,k} = \sum_{j<k} \hat{H}_{j,k+1} = \sum_j \hat{H}_{j,j+1} (\hat{P}_k, \ldots) \) for an integrable vertex model.

(iii) Recently, Kauffman and Saleur have found a relation between the \( R \)-matrix for the simplest quantum superalgebra \( U_q\mathfrak{sl}(1,1) \) and the free fermions “propagating” on a knot diagram \[8\]. We hope further application of the new \( R \)-matrices in this Letter can lead to the generalization of their theory for the higher-rank quantum superalgebras or the non-generic case that \( q \) is a root of unity.

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