Representations of quantum matrix algebra $M_q(2)$ and its $q$-boson realization

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In this paper it is proved that the nontrivial finite-dimensional irreducible representations of the quantum matrix algebra $M_q(2)$ (or the coordinate ring of the quantum group $GL_q(2)$) exist only when $q$ is a root of unity, and thereby construct such representations for $M_q(2)$. Some finite-dimensional indecomposable representations of $M_q(2)$ are also explicitly presented. Finally, the $q$-boson realization of $M_q(2)$ is discussed, and it is shown that it works in this case as well as in the quantum algebra case.

I. INTRODUCTION

Associated with nonlinear physical problems through the Yang–Baxter equation, quantum group, and quantum algebra theory has drawn much attention from both mathematical and theoretical physics fields. Up to now, there have been a great number of papers dedicated to the representation theory of quantum universal envelope algebras (quantum algebras) and many elegant results have been obtained. On the other hand, for quantum groups, it seems that people mainly focus on the study of their comodules, and that the other aspect, the representation theory of their coordinate rings, or what we call quantum matrix algebras in this paper, is ignored. As far as we know, there is still no systematic discussion on the structure of the representations of quantum matrix algebras. What has been done is just the construction of some concrete representations.

In this paper we will make systematic investigation of the representation of $M_q(2)$, the quantum matrix algebra of the $q$ quantum group $GL_q(2)$. Our main result is expressed in two propositions in Sec. III, which assert that only when $q$ is a root of unity will the nontrivial finite-dimensional irreducible representations exist, and in this case, the dimension of the representations is either $p$ or $p/2$. In the process of the proof of this result, we explicitly demonstrate the construction of the cyclic representations and the “highest weight” representations. Based on the discussion made in Sec. III, some indecomposable representations are constructed in Sec. IV. Finally, a $q$-boson realization of the quantum matrix algebra is given.

II. RELATIONS ON THE QUANTUM MATRIX ALGEBRA $M_q(2)$

Let us begin with some basic concepts.

Quantum group $GL_q(2)$ is a set of matrices $T$’s that satisfy the equation

$$\tilde{R}(T \otimes T) = (T \otimes T)\tilde{R},$$

(2.1)

where

$$\tilde{R} = \begin{bmatrix} q & 0 & 1 \\ 0 & 1 & q^{-1} \\ 1 & q & q \\ \end{bmatrix}.$$ 

Its coordinate ring, the quantum matrix algebra $M_q(2)$, is defined as a $c$ algebra generated by the elements $a, b, c,$ and $d$ with the relations

$$ab=qba, \quad ac=qca,$$

$$bd=qdb, \quad cd=qdc,$$

$$ad-da=(q-q^{-1})bc, \quad bc=cb,$$

(2.2)

which follow from Eq. (2.1), if we write

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

Using the above relations one can easily verify that the quantum determinant

$$D_q=\det_q T = ad-qbc$$

belongs to the center of $M_q(2)$.

Before passing to the next section, we give two lemmas, which can be proved by direct calculation.
Lemma 1:
\[
ad^n = a^n + q^{-1}(q^{2n} - 1)d^{n-1}bc, \\
da^n = a^n d + q(q^{2n} - 1)a^{n-1}bc,
\]
for \(n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots \} \).

Lemma 2: When \(q\) is a root of unity, i.e., \(q^2 = 1\), \(a, b, c, d\) belong to the center of \(M_q(2)\).

These two results will be used in the subsequent sections.

III. FINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS

To simplify the discussion we first distinguish between trivial and nontrivial \(M_q(2)\) modules.

Definition 3.1: The action of an operator is called to be trivial if its kernel is the whole space that it acts on.

Definition 3.2: An \(M_q(2)\) module \(V\) is called a trivial module if at least one of the generators of \(M_q(2)\) acts trivially on \(V\); otherwise, it is called a nontrivial module.

Remark: The study of a trivial \(M_q(2)\) module is much easier than that of a nontrivial one. In fact, the structure of a trivial \(M_q(2)\) module collapses into the structure of a module of a simpler algebra. For example, when the action of the element \(b\) is trivial, one only needs to investigate the module of the algebra generated by \(a, c, d\) with the relations
\[
[a, d] = 0, \quad ac = qa, \quad cd = qdc.
\]

For this reason, we mainly study nontrivial \(M_q(2)\) modules in this paper.

Now we are in the position to state the following propositions.

Proposition 1: A finite-dimensional irreducible \(M_q(2)\) module \(V\) is trivial when at least one of the following two sets contains nonzero vectors:
\[
K_b = \{v \in V | bv = 0\}, \quad K_c = \{v \in V | cv = 0\}.
\]

Proof: There are several cases to consider.

Case 1: \(K_b \neq \{0\}, K_c \neq \{0\}, \) and \(K_b \cap K_c \neq \{0\}\). This condition leads to the result
\[
K = \{v \in V | bv = cv = 0\} = \{0\}.
\]

On the other hand, it is evident that
\[
aK \subset K, \quad dK \subset K, \quad [a, d]K = (q - q^{-1})bcK = 0.
\]

So there must exist a nonzero vector \(v_0 \in K\), such that
\[
av_0 = \lambda_a v_0, \quad dv_0 = \lambda_d v_0 \quad (\lambda_a, \lambda_d \in \mathbb{C}).
\]

It means that \(C_{v_0}\) is stable under the action of \(M_q(2)\). But \(V\) is an irreducible \(M_q(2)\) module; consequently we have
\[
V = C_{v_0}, \quad bV = cV = 0.
\]

Case 2: \(K_b \neq \{0\}, K_c = \{0\}\), and \(K_b \cap K_c = \{0\}\).

In this case, one can find a nonzero vector \(v_0 \in K_b\) satisfying \(cv_0 = \lambda v_0 (\lambda \neq 0)\) due to the condition \(K_b \cap K_c = \{0\}\) and the obvious fact \(cK_b \subset K_b\). When \(q^2 = 1\), according to Schur's lemma and Lemma 2, we have
\[
d^\epsilon = \xi_d, \quad d^\epsilon = \xi_d \quad (\xi_d, \xi_c \in \mathbb{C}),
\]

thanks to the irreducibility of \(V\). Thus the vector space
\[
S = \text{span}\{a^m d^n v_0 | 0 < m, n < \rho\},
\]
is a submodule of \(V\), and hence \(S = V, \ bV = 0\). When \(q\) is generic, let us consider the sequence
\[
v_0, dv_0, d^2 v_0, \ldots.
\]

Using the relation \(cd = q dc\), one gets
\[
c(d^i v_0) = \lambda q^i d^i v_0, \quad i = 0, 1, 2, \ldots.
\]

That is to say that if \(d^i v_0 \neq 0\), it is an eigenvector of \(c\) corresponding to the eigenvalue \(\lambda q^i\). Under the condition that \(q\) is not a root of unity, \(\lambda q^i \neq \lambda q^j\) if \(i \neq j\), so one comes to the conclusion, because \(V\) is finite dimensional, that there exists some \(i \in \mathbb{Z}^+\) such that \(\{d^i v_0 | i = 0, 1, \ldots, i-1\}\) are linearly independent, and \(d^i v_0 = 0\). Denote \(v_0 = d^{i-1} v_0\); then similarly, there is some \(m \in \mathbb{Z}^+\) to guarantee the linear independence of the vectors \(\{d^i v_0 | i = 0, 1, \ldots, m - 1\}\) and the condition \(d^m v_0 = 0\). As a result, the vector space
\[
S' = \text{span}\{a^i d^n v_0 | i = 0, 1, \ldots, m - 1\}
\]
is invariant, with respect to the action of \(M_q(2)\). Thus, from the irreducibility of \(V\), it follows that \(S' = V\), and one finally gets \(bV = bS' = 0\), which is what one wants to prove.

Case 3: \(K_b = \{0\}, K_c \neq \{0\}\) or \(K_b \neq \{0\}, K_c = \{0\}\).

The discussion is the same as in case 2.

Proposition 2: If there exists a nontrivial finite-dimensional irreducible \(M_q(2)\) module, \(q\) must be a root of unity.

Proof: Let \(V\) be a nontrivial finite-dimensional irreducible \(M_q(2)\) module; then from Proposition 1 we have \(K_b = K_c = \{0\}\). Therefore there exists such a nonzero vector \(v_0 \in V\) that satisfies
\[
bv_0 - \lambda v_0, \quad cv_0 - \lambda v_0 \quad (\lambda_b, \lambda_c \neq 0),
\]

because \(b\) and \(c\) commute. Suppose \(q\) is not a root of unity; then it follows from the argument given in the proof of Proposition 1 (case 2) that one can find a nonzero vector \(u_0\), satisfying \(du_0 = 0\) (in fact, \(u_0\)
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$d^{-1} v_0 \in \mathbb{Z}^+$ and $d^f u_0 = 0$, but $d^{-1} u_0 \neq 0$ for some $f \in \mathbb{Z}^+$. Using Lemma 1, one has

$$0 = d^{-1} u_0 = q^{-2f} (q^{-2f} - 1) \lambda_{q^f} \lambda_{q^{-2f}} d^{-1} u_0.$$ 

It requires $q^{-2f} = 1$, which contradicts the presumption.

**Remark:** It is easy to see that the irreducibility condition is not absolutely necessary. In fact, condition (3.1) is essential. In other words, if there exists a finite-dimensional $\mathcal{M}_q(2)$ module in which one can find a nonzero vector satisfying Eq. (3.1), then one can prove that $q$ must be a root of unity.

Thanks to Proposition 2, for nontrivial finite-dimensional irreducible representations, we only need to consider the case that $q$ is a root of unity. So from now on in this section we will suppose $q^p = 1$, where $p$ is the smallest integer satisfying this condition.

**Proposition 3:** There exist only $p$-dimensional and $p/2$-dimensional nontrivial irreducible $\mathcal{M}_q(2)$ modules besides infinite-dimensional modules.

**Proof:** Let $V$ be a nontrivial finite-dimensional irreducible $\mathcal{M}_q(2)$ module; then Schur's lemma is available. In view of Lemma 2, we have

$$D_v = \eta, \quad d^0 = \eta_m, \quad d^p = \eta_m, \quad \eta_{m} \neq 0,$$

and from the proof of Proposition 2 we know that there exists a nonzero vector $v_0 \in V$ satisfying

$$d v_0 = \lambda d v_0, \quad \eta v_0 = \lambda \eta v_0, \quad \lambda, \eta \neq 0.$$

For convenience, in the following we divide the proposition into two parts and prove them separately.

1. If $\eta \neq 0$ or $\eta \neq 0$, then dim $V = p$.

Without losing the generality, we take $\eta \neq 0$. Let us first exclude the possibility that dim $V < p$. If dim $V < p$, then by considering the sequence $\{d^j v_0, j = 0, 1, 2, \ldots\}$ one gets $d^j v_0 = 0 (j < p)$, and hence $\eta d v_0 = d^p - (d v_0) = 0$. This contradicts the assumption that $\eta \neq 0$. Next, we assume dim $V > p$. For the same reason, we have $d v_0 = 0$ if $i < p$. Thus the vectors $\{v_0, d v_0, d^2 v_0, \ldots, d^{p-1} v_0\}$ are linearly independent. We will show that they span a submodule. In fact, after some direct calculation, we get

$$d v_0 = v_{n+1},$$

$$d v_{p-1} = \eta d v_0,$$

$$d v_{p-1} = d^p v_0 + q^{-1} (q^2 - 1) \lambda \eta d^p v_0,$$

$$d v_0 = \eta^{-1} \lambda \eta d^p v_0,$$

$$b v_n = \lambda \eta d^n v_0,$$

$$c v_n = \lambda \eta d^n v_0.$$

Thus, from the irreducibility condition we have dim $V = p$. Considering the equation

$$0 = d^m v_0 = \lambda \eta d^m v_0,$$

we obtain $m = p/2$ in the case that $p$ is even. It is only routine to verify that in this case (3.3) indeed defines a $p/2$-dimensional irreducible representation, which we like to call a “highest weight” representation. In a similar way, one can easily convince oneself that when dim $V > p$, there exists a $p$-dimensional or a $p/2$-dimensional irreducible submodule of $V$. This completes our proof.
IV. FINITE-DIMENSIONAL INDECOMPOSABLE REPRESENTATIONS

In this section we aim to construct some finite-dimensional indecomposable representations of $M_q(2)$. Let us consider the $M_q(2)$ analog of Verma space,

$$W = \text{span}\{f(n) = d^n|\lambda\rangle | b|\lambda\rangle = q^{1/2}\lambda\},$$

where $c|\lambda\rangle = q^{1/2}|\lambda\rangle, a|\lambda\rangle = 0$, which carries an infinite-dimensional representation $\rho^1$:

$$df(n) = f(n+1),$$

$$af(n) = q^{-1} + t_1 + t_2(q^n-1)f(n-1),$$

$$bf(n) = q^{1/2} + sf(n),$$

$$Df(n) = q^{1/2} - t_{-1}f(n).$$

The last equation shows that to some extent these kind of representations distinguish themselves from one another by the value of the central element $D$. It is obvious that when $q$ is generic the representation $\rho^1$ is irreducible. But if $q^2 = 1$, one gets the invariant subspaces

$$W_{K,p} = \text{span}\{f(Kp), f(Kp+1), \ldots\}, \quad K = 1, 2, \ldots,$$

owing to the equation $af(Kp) = 0$. Evidently, one has

$$W_{1,p} \supset W_{2,p} \supset W_{3,p} \supset \cdots$$

and

$$Q_{1,p} \subset Q_{2,p} \subset Q_{3,p} \subset \cdots,$$

where $Q_{K,p} = W/W_{K,p}$ is the quotient space corresponding to $W_{K,p}$:

$$Q_{K,p} = \text{span}\{f(n) = f(n) \text{Mod } W_{K,p} | 0 < n < Kp-1\}.$$

On each $Q_{K,p}$, $\rho^1$ induces a $Kp$-dimensional representation $\rho^1_{K,p}$:

$$d\bar{f}(n) = \bar{f}(n+1),$$

$$d\bar{f}(Kp-1) = 0,$$

$$a\bar{f}(n) = q^{-1} + t_1 + t_2(q^n-1)\bar{f}(n-1),$$

$$b\bar{f}(n) = q^{1/2} + s\bar{f}(n),$$

$$c\bar{f}(n) = q^{1/2} + f(n),$$

for which we have the following.

Proposition 4: Suppose $p$ is odd; then

1. when $K = 1$, $\rho^1_{K,p}$ is irreducible; and
2. when $K > 1$, $\rho^1_{K,p}$ is reducible, but not completely reducible.

Proof: (1) The first case.

If $\rho^1_{K,p}$ were reducible, there would exist a nontrivial invariant subspace $W_\notin Q_{1,p}$. Most generally, an element $x \in W_\notin$ can be written as $x = \sum_{i=0}^{(K-1)p-1} c_i f(i)$. Let $l$ be the smallest integer that satisfies $c_l \neq 0$. Then we have $d^{(K-1)p-1}x = c_i f(p-1)$ and as a result, $\{a d^{(K-1)p-1}x | i = 0, 1, 2, \ldots, p-1\}$ constitute a basis of $Q_{1,p}$. It is a contradiction.

(2) The second case.

First of all, we point out that $\rho^1_{K,p}$ is completely reducible if and only if for each invariant subspace $W \subset Q_{K,p}$ there is an invariant subspace $W$ complementary to it. Now let us consider the subspace

$$W = \text{span}\{f(n) | (K-1)p < n < Kp-1\},$$

It is easily seen that $W$ is invariant under $M_q(2)$. One can prove that it does not have an invariant complementary subspace $\bar{W}$ in $Q_{K,p}$. Otherwise, a nonzero element $x \in \bar{W}$ can be written as

$$x = \sum_{i=0}^{(K-1)p-1} c_i f(i) + \sum_{j=(K-1)p}^{Kp-1} d_\bar{f}(j),$$

where there is at least one nonzero $c_i$. Let $l$ be the smallest integer, such that $c_l \neq 0$. Then we have

$$d^{(K-1)p-1}x = \sum_{i=0}^{(K-1)p-1} c_i f(i + (K-1)p) + \sum_{j=(K-1)p}^{Kp-1} d_\bar{f}(j + (K-1)p),$$

where $f(m) = 0$ when $m \notin Kp$. Obviously, $d^{(K-1)p-1}x \in \bar{W}$. This contradicts the assumption that $W$ is invariant.

Remark 1: One easily sees that we have, in fact, proved that for any of the invariant subspaces,

$$W_i = \text{span}\{f(n) | (K-1)p < n < Kp-1\}, \quad i = 1, 2, \ldots, K-1,$$

in $Q_{K,p}$ there is no invariant subspace complementary to it.

Remark 2: When $p$ is an even integer, one can make similar discussion.

V. $q$-BOSON REALIZATION OF $M_q(2)$

This section is devoted to showing that the $q$-boson realization theory of quantum algebras also works for the quantum matrix algebra.

As usual, denote by $B_q$ the $q$-boson algebra generated by the elements $A^+$, $Q^+$ and the relations...
A q-boson realization of $M_q(2)$ is then defined to be the image of a homomorphic map $B$ from $M_q(2)$ to $B_q$.

Suppose $W = \text{span}\{ f(n) | n = 0, 1, 2, ... \}$ is a representation space of $M_q(2)$ and $\psi$ is the natural isomorphism from $W$ to the $q$-Fock space

$$\mathcal{F}_q = \text{span}\{ F(n) = A^+n|0\} | A^-|0\} = Q^-, \quad n = 0, 1, 2, ... \},$$

i.e., $\psi$ is a linear map satisfying

$$\psi f(n) = F(n),$$

we define the induced homomorphism $\psi^* = \text{End}(W) \rightarrow \text{End}(\mathcal{F}_q)$ by

$$\psi^*(g) = \psi \cdot g \cdot \psi^{-1}, \quad \forall g \in \text{End}(W).$$

Now, we are prepared to present the following.

**Proposition 4:** Let $\rho_F$ be the natural representation of $B_q$ on $\mathcal{F}_q$ and $\rho$ be a representation of $M_q(2)$. Then, if the diagram

$$\begin{array}{ccc}
M_q(2) & \longrightarrow & B_q \\
\rho & \downarrow & \rho_F \\
\text{End}(W) & \longrightarrow & \text{End}(\mathcal{F}_q)
\end{array}$$

is commutative, $B(M_q(2))$ is a q-boson realization of $M_q(2)$.

**Proof:** immediate.

Before concluding this paper, let us consider an example. Taking $\rho$ to be the representation given by (4.1), by direct calculation we obtain

$$B(a) = q^{-1+\lambda_1+\lambda_2}(q-q^{-1})A^-Q^+, \quad (5.2)$$

$$B(b) = q^1Q^+, \quad B(c) = q^2Q^+, \quad B(d) = A^+.$$

It is easy to verify that they indeed satisfy the basic relations in (2.2). Certainly, using (5.2), we can discuss the representations of $M_q(2)$ on $\mathcal{F}_q$ but we will not do it here.

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