A new q-deformed boson realization of quantum algebra sl_q(2) and nongeneric sl_q(2) R-matrices

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In this paper a new q-deformed boson realization of quantum algebra sl_q(n + 1) is presented for the first time, and its representations are obtained in the nongeneric case that q is a root of unity. A new type of sl_q(2) R-matrices are systematically constructed through the universal R-matrix.

I. INTRODUCTION

It is well known that R-matrices for Yang–Baxter equation play a crucial role in nonlinear physics such as low-dimensional integrable field theory, exactly solvable models in statistical mechanics, and conformal field theory.1–3 The standard way of obtaining an R-matrix is substituting the representation of a quantum group (sometimes strictly called q-analog of universal enveloping algebra of a classical Lie algebra or quantum algebra) into the so-called universal R-matrix.4–6 The R-matrices obtained in this way are called standard R-matrices. Recently, many new R-matrices, that are sometimes called nonstandard R-matrices, have been constructed by means of the extended Kauffman’s diagram technique.7 These new R-matrices associated with classical Lie algebras provide new representations of braid group and have been Yang–Baxterized to satisfy Yang–Baxter equation with a spectral parameter.8,9 The possible relations between the nonstandard R-matrices and quantum algebras have also been discussed with the sl_q(2) case as an example.10

The main purpose of this paper is to construct a new type of R-matrix essentially different from the standard and nonstandard ones in the standard way. To this end, we first establish a nongeneric q-deformed boson realization of the finite-dimensional representations of quantum algebra sl_q(2) based on the previous works,11–14 then we generally construct a new class of R-matrices associated with sl_q(2) through the universal R-matrix. Because of the indecomposi- tion property of the new representations used in this paper, the obtained R-matrices possess the following nongeneric properties: (1) They satisfy the Yang–Baxter equation only when q is a root of unity and cannot be obtained from the corresponding standard or nonstandard R-matrices by letting q^2 = 1. (2) They no longer have the eigenvalue structure that the standard and nonstandard R-matrices possess, thus the scheme to Yang–Baxterize them may be completely new and need further investigation.

II. A NEW q-DEFORMED BOSON REALIZATION OF sl_q(n+1)

According to the discussion about the q-deformed boson operators independently presented by different authors, we can define a q-deformed boson algebra B_q(I) as an associative algebra over the complex number field. It is generated by a_i^+, a_i, N_i (i = 1, 2, ..., n) that satisfy

\[ [a_i^+, a_j] = \delta_{i,j}, \quad [N_i, a_j] = \pm \delta_{i,j} a_j^\pm, \]

where \( \delta_{i,j} \) is the Kronecker delta function. From (2.1) we can easily prove that the following elements of \( B_q(I+1) \)

\[ x_i^+ = N_i a_i, \quad x_i^- = a_{i+}^-, \]

indeed define a q-deformed boson realization of sl_q(n + 1), that is to say, (2.1) satisfy (2.2).

\[ X_i^+, X_i^- \] dead in [H_j], 1

X_i^+ X_i^+ = X_i^- X_i^- = X_i^+ X_i^- + X_i^- X_i^+ = 0,

where \( \alpha_q = 2\delta_q - \delta_{q+1} - \delta_{q-1} \). From (2.1) we can easily prove that the following elements of \( B_q(n+1) \)

\[ X_i^+ = a_i^+, \quad X_i^- = a_{i+1}^-, \]

indeed define a q-deformed boson realization of sl_q(n + 1), that is to say, (2.3) satisfy (2.2).

Before going further we would like to point out that there is an essential difference between the above realization
and the previous one. The previous q-deformed boson realization \( B(\mathfrak{sl}_q(n+1)) \): \( \hat{g} = B(g) \) satisfies

\[
[\hat{g}, N] = 0, \quad N = \sum_{i=1}^{n+1} N_i \quad \forall g \in \mathfrak{g} \mathfrak{l}_q(n+1).
\]

In other words, "the number of particles" remains unchanged under the action of \( \hat{g} \), but it is not the case now. In contrast, there exist some \( g \in \mathfrak{g} \mathfrak{l}_q(n+1) \) such that \( [\hat{g}, N] \neq 0 \), so it is no longer possible to find a finite-dimensional representation space for \( \mathfrak{g} \mathfrak{l}_q(n+1) \) when \( q \) is generic. Fortunately, when \( q \) is a root of unity we can find some \( \mathfrak{g} \mathfrak{l}_q(n+1) \)-invariant subspaces and finite-dimensional representation spaces for \( \mathfrak{g} \mathfrak{l}_q(n+1) \). This is done in the next section.

### III. FINITE-DIMENSIONAL REPRESENTATIONS OF \( \mathfrak{g} \mathfrak{l}_q(n+1) \)

For the q-deformed boson algebra \( \mathfrak{g} \mathfrak{l}_q(n+1) \), we define a q-deformed Fock space \( \mathcal{F}_q(n+1) \):

\[
\{|m\} = |m_1, m_2, \ldots, m_{n+1}\rangle = d_1^{m_1} d_2^{m_2} \cdots d_{n+1}^{m_{n+1}} |0\rangle
\]

where the q-vacuum state \( |0\rangle \) satisfies

\[
a_i |0\rangle = N_i |0\rangle = 0, \quad i = 1, 2, \ldots, n + 1,
\]

then we obtain a natural representation \( \rho \) of \( \mathfrak{g} \mathfrak{l}_q(n+1) \) on \( \mathcal{F}_q \):

\[
g_i |m\rangle = |m_1 + \delta_i + \delta_i, m_2, \ldots, m_{n+1}\rangle,
\]

\[
f_i |m\rangle = - (m_1 + 1) |m_1, m_2, \ldots, m_{n+1}\rangle,
\]

\[
h_i |m\rangle = \pm (m_1 + m_2, \ldots, m_{n+1} + 1)|m\rangle,
\]

where \( g_i = a_i^* a_i^+ + 1 \), \( f_i = - a_i a_i^+ + 1 \), and \( h_i = \pm (N_i + N_i + 1 + 1) \) are the elements of the set \( \{X^+, H, J_i, i = 1, 2, \ldots, n + 1\} \). From (3.1) it is easily seen that there is a \( \rho \)-invariant subspace \( V_{n+1}^{(n+1)} \) (for a given \( N \in \mathbb{Z}^+ \))

\[
\left\{ |m\rangle \in \mathcal{F}_q(n+1) \mid \sum_{k=1}^{n+1} m_{2k} \leq N \right\}.
\]

When \( q \) is nongeneric, i.e., \( q^2 = 1 \), \( V_{n+1}^{(n+1)} \) has \( \rho \)-invariant subspaces

\[
W_{n+1}^{(n+1)}(\alpha, \rho) \mid \{l = 1, 2, \ldots, n+1 \mid (n-1)(n-1)^{l-1} \alpha, \xi \in \mathbb{Z}^+ \}
\]

It will be shown that in some cases, the quotient spaces \( V_{n+1}^{(n+1)} / \Sigma W_{n+1}^{(n+1)}(\alpha, \rho) \) are finite dimensional.

### IV. NONGENERIC REPRESENTATIONS OF \( \mathfrak{g} \mathfrak{l}_q(2) \)

In order to construct nongeneric \( R \)-matrices associated with quantum algebra \( \mathfrak{g} \mathfrak{l}_q(2) \), in this section we will study the nongeneric representations of \( \mathfrak{g} \mathfrak{l}_q(2) \) in detail.

Using the new q-deformed boson realization of \( \mathfrak{g} \mathfrak{l}_q(2) \), we can obtain a representation \( \rho^{(n+1)} \).
\[ J_+ f_N(m) = f_N(m + 1), \]
\[ J_- f_N(m) = -[m]_N f_N(m - 1), \]
\[ J_0 f_N(m) = (2m + 1) f_N(m), \]  

on the invariant subspace \( W_N^\alpha (\alpha \in \mathbb{Z}^+): \)
\[
\{ f_N(m) = (a_1^* a_2^* a^{N+1} m)_{\in \mathcal{F}} \ | \ m + N \geq \alpha p \},
\]

of the \( q \)-deformed Fock space \( \mathcal{F} (2): \)
\[
\{ |m, m_2 \rangle = a_1^* a_2^{m_1} a^{m_2} |0 \rangle, |N_1 \rangle = a |0 \rangle = 0, \]
\[
i = 1, 2, m_1, m_2 \in \mathbb{Z}^{+}. \}

It is obvious that this representation is infinite dimensional. To obtain a finite-dimensional representation let us consider the quotient space \( Q_\alpha (n, N) = \mathcal{F} (2)/W_N^\alpha (N = \alpha p - n): \)
\[
\{ f_N(m) = (J, M) \in \mathcal{F} (2)/W_N^\alpha (\alpha p - n) \}.
\]

It should be pointed out that because in the nongeneric case \( [M + p] = [M] \), we cannot obtain any new representations on \( W_N^\alpha (N \geq \alpha p) \). Moreover, we can prove that all of the above representations are indecomposable, so we can expect that they will give rise to a new type of R-matrix.

V. NONGENERIC \( R \)-MATRICES ASSOCIATED WITH \( \text{sl}_q (2) \)

Having constructed the nongeneric representations of \( \text{sl}_q (2) \), now we are prepared to obtain the nongeneric \( R \)-matrices associated with them. First, we rewrite (4.2) and (4.3) as
\[
(J_+)_m = \delta_m^{\alpha p} (J - m),
\]
\[
(J_-)_m = [J + m] [J + m + N] \delta_m, \quad m, m' = -J, -J + 1, \ldots, J,
\]
and
\[
(J_0)_m = (2m + \alpha p + N) \delta_m, \quad m, m' = -J, \ldots, J, \]

which are convenient to use. Then, we substitute them into the universal R-matrix
\[
R = q^{(1/2)} (J_+ J_0 J_-) \sum_n \frac{1 - q^{-2}}{[n]_q} \times q^{-n(n-1)/2} q^{m' - m'} \times \prod_{i=1}^{n} [J + l + i]_q [J + 1 - (l + i)]_q \delta_{m+n}^{\alpha p} \delta_{l-n}^{\alpha p} \quad (5.1)
\]
and
\[
R = q^{(1/2)} (2m' + \alpha p + N) (2l' + \alpha p + N) \delta_m^{\alpha p} \delta_l^{\alpha p} + q^{(1/2)} (2m' + \alpha p + N) (2l' + \alpha p + N) \sum_{n=1}^{2} \frac{1 - q^{-2}}{[n]_q} \times q^{-n(n-1)/2} q^{m' - m'} \times \prod_{i=1}^{n} [J + l + i]_q [J + 1 - (l + i)]_q \delta_{m+n}^{\alpha p} \delta_{l-n}^{\alpha p} \quad (5.2)
\]
respectively. The R-matrices concerning two different representation spaces can be obtained in the same way.

To show that (5.1) and (5.2) can provide some new R-matrices essentially different from the standard and nonstandard ones already known to us, we give an example here. Taking \( \alpha = 1, \ p = 3, \) and \( N = 2 \) in (5.2), we have

\[ J_+ f_N/(J, M) = f_N/(J, M + 1) \theta(J - M), \]
\[ J_- f_N/(J, M) = -[J + M] [J + M + N] f_N/(J, M - 1), \]

\[ J_0 f_N/(J, M) = (2M + \alpha p + N) f_N/(J, M), \]
Finally, we would like to point out that the new $R$-matrices possess nongeneric properties as mentioned in the Introduction, so many problems naturally arise. Further discussions about them are beyond the scope of this paper, and will be presented in forthcoming papers.

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$R^{11} = \begin{pmatrix}
q^{1/2} & q^{1/2}(q - q^{-1}) & 0 \\
q^{-1/2} & 0 & q^{1/2}(q - q^{-1}) \\
0 & q^{-3/2} & q^{-3/2} \\
0 & 0 & q^{3/2}
\end{pmatrix}, \quad q^3 = 1.$


14. C. P. Sun and M. L. Ge, “The $q$-deformed boson realization of representations of quantum universal enveloping algebra for $q$ a root of unity (I) and (II),” preprint NIM-TP20; C. P. Sun, K. Xue, X. F. Liu, and M. L. Ge, “The boson realization of non-generic $sl_q(2)$-$R$-matrices for Yang-Baxter equation,” preprint NIM-TP3.