Inhomogeneous boson realization of indecomposable representations of Lie algebras

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By making use of the differential realization of Lie algebras in the space of inhomogeneous polynomials of a certain number of variables, the corresponding inhomogeneous boson realization of Lie algebras is given. A new kind of indecomposable representations of Lie algebras are studied on the universal enveloping algebra of Heisenberg–Weyl algebra, its subspaces and its quotient spaces. The finite-dimensional representations are naturally obtained on the subspaces of Fock space. As an example, the indecomposable and irreducible representations of the Lie algebra su(2) are discussed in detail.

I. INTRODUCTION

Indecomposable representations of physically relevant Lie algebras have been suggested for the description of unstable particles. 1,2 Gruber and his co-workers studied the indecomposable representations of Lie algebras on the universal enveloping algebra of this Lie algebra by making use of the purely algebraic method. 3,4 One of the authors has studied the indecomposable representations of Lie algebras on the universal enveloping algebra of Heisenberg–Weyl algebra, its subspaces and its quotient spaces by making use of the homogeneous boson realization (HBR) of Lie algebras. 5–7 In this paper we will study the indecomposable representations of Lie algebras by making use of the inhomogeneous boson realization (IHBR) of Lie algebras, which is obtained from the corresponding inhomogeneous differential realization (IHDR).

However, the IHDR of Lie algebras itself is very useful in “quasi-exactly-solvable problems of quantum mechanics” (QESP) discovered recently. 8–11 QESP have been proved to be related to the IHDR of Lie algebras. Turbiner has studied the one-dimensional QESP by using the IHDR of sl(2) algebra, 9 and pointed out that a similar procedure for the search of multidimensional QESP can be developed if we use the IHDR of sl(m) algebra. 8 Shifman and Turbiner studied the two-dimensional QESP by making use of the IHDR of su(2) × su(2), so(3), and su(3) algebras. 10 Therefore, the IHDR of Lie algebras given in this paper will play an important role in the search for multidimensional QESP.

In this paper, the IHDR of Lie algebras is generated by generalizing Shifman’s discussions in Ref. 11. The corresponding IHBR of Lie algebras is obtained with provision for the corresponding relation between differential operators and creation and annihilation operators of boson states. In comparison with the HBR given in Refs. 5–7, the IHBR of Lie algebras uses creation and annihilation operators less than the HBR, and enables us to obtain the finite-dimensional representations on the subspaces of Fock space, while in Refs. 5–7 we can only obtain the finite-dimensional representations on the quotient spaces of Fock space. By making use of the IHBR of Lie algebras, a new class of indecomposable representations of Lie algebras are studied on the universal enveloping algebra of Heisenberg–Weyl algebra, its subspaces and its quotient spaces. As an explicit example, the indecomposable representations of su(2) algebra are studied in detail.

Although the IHDR of Lie algebras given in this paper looks trivial, we can use its corresponding IHBR to obtain various nontrivial indecomposable representations on the universal enveloping algebra of Heisenberg–Weyl algebra, its subspaces and its quotient spaces.

The symbols Z + and N denote the set of non-negative integers and the set of positive integers, respectively. The symbol C denotes the field of complex numbers.

II. FROM IHDR TO IHBR

A. IHDR of Lie algebras

Let the basis for the Lie algebra L be all \( \{ T_p \} \) that satisfy the Lie product \( [T_p, T_q] = \sum_r C_{pq}^r T_r \), where \( C_{pq}^r \) are structure constants. Let \( D \) be a faithful representation of \( L \) with dimension \( m < \infty \), and let \( \{ e_1, e_2, \ldots, e_m \} \) be the basis for a representation space. Then we have

\[
T_p e_i = \sum_{j=1}^m D(T_p)_{ji} e_j .
\]  (2.1)

Since we would like to construct a realization on polynomials, we introduce \( m \) independent variables \( \{ x_1, x_2, \ldots, x_m \} \) and identify them with the basis vectors \( \{ e_1, e_2, \ldots, e_m \} \):

\[
x_i \leftrightarrow e_i \quad (i = 1, 2, \ldots, m) .
\]  (2.2)

Now Eqs. (2.1) and (2.2) imply

\[
T_p x_i = \sum_{j=1}^m D(T_p)_{ji} x_j .
\]  (2.3)

Equation (2.3) immediately allows us to write the \( T_p \) in the differential form

\[
T_p = \sum_{i=1}^m D(T_p)_{ji} x_j \frac{\partial}{\partial x_i} .
\]  (2.4)

This realization is obviously valid not only for the first-order homogeneous polynomial space with basis \( \{ x_1, x_2, \ldots, x_m \} \) that carries the representation \( D \) but also for the space of \( n \)-th order homogeneous polynomials spanned by

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\[
\begin{align*}
\left\{ x_1^i x_2^j \cdots x_m^k \right\} & \sum_{k=1}^m i_k = n, \ i_k \in \mathbb{Z}^+ \} , \\
\end{align*}
\] which carries the symmetrized direct product representation
\[
D_s^* = \left( D \otimes D \otimes \cdots \otimes D \right) \text{ symmetrized} .
\] (2.6)

In fact, the following equation defines a new realization \( \hat{T}_p \) on the space of nth-order homogeneous polynomials:
\[
\hat{T}_p (x_1^i x_2^j \cdots x_m^k) = \sum_{r=1}^n x_1^i \cdots x_{r-1}^j (T_p x_r^i) x_{r+1}^i \cdots x_m^k .
\] (2.7)

From Eqs. (2.4) and (2.7) it follows that
\[
\hat{T}_p = \sum_{i=1}^m D(T_p) x_j \frac{\partial}{\partial x_j} \equiv T_p .
\] (2.8)

We call the realization (2.8) on the space of homogeneous polynomials the homogeneous differential realization (HDR) that corresponds to the HBR given in Refs. 5-7.

However, we would like to construct the differential realization on the space of inhomogeneous polynomials for the needs of QESP. For this purpose we define a new variable:
\[
\xi_i = \frac{x_i}{x_m} \ (i = 1,2,\ldots,m-1) ,
\] (2.9)
\[
x_1^i x_2^j \cdots x_m^k
\]
\[
\Rightarrow \xi_1^i \xi_2^j \cdots \xi_{m-1}^k x_m^k \left( \sum_{k=1}^{m-1} i_k = 0,1,2,\ldots,n \right) .
\] (2.10)

Then the linear space \( \mathcal{P} \) spanned by
\[
\mathcal{P} : \left\{ \xi_1^i \xi_2^j \cdots \xi_{m-1}^k x_m^k \left| \sum_{k=1}^{m-1} i_k = 0,1,\ldots,n, \ i_k \in \mathbb{Z}^+ \right. \right\}
\] (2.11)
is a space of inhomogeneous polynomials with dimension
\[
\dim \mathcal{P} = \sum_{k=0}^{n} \frac{(m+k-2)!}{k! (m-2)!} .
\] (2.12)

In fact, we can regard \( \{ \xi_i \ | \ i = 1,2,\ldots,m-1 \} \) as the local coordinates of the projective space
\[
PR^{m-1} : \{ [x] | [x] = \{ y \in \mathbb{R}^n \ | \ y = \lambda x, \ \lambda \in \mathbb{R} \} \}
\]
of
\[
\mathbb{R}^n : \{ x = [x_1,\ldots,x_m] | x_1,\ldots,x_m \in \mathbb{R} \},
\]
where \( \mathbb{R} \) is the field of real numbers. The space \( \mathcal{P} \) is the polynomial space with regard to the local coordinates \( \{ \xi_i \} \) of \( PR^{m-1} \) through the corresponding relation (2.10) between the basis for \( \mathcal{P} \) and the basis for the space of homogeneous polynomials.

It is easy to deduce that
\[
\hat{T}_p \ x_m^n = nx_m^n D(T_p) x_m^{n-1} + \sum_{i=0}^{n-1} D(T_p) x_m^{n-1} \frac{\partial}{\partial x_m} \xi_i ,
\]
\[
\hat{T}_p \ x_m^n = nx_m^n D(T_p) x_m^{n-1} + \sum_{i=0}^{n-1} D(T_p) x_m^{n-1} \frac{\partial}{\partial x_m} \xi_i .
\]
\[
\Rightarrow \xi_1^i \xi_2^j \cdots \xi_{m-1}^k x_m^k \left( \sum_{k=1}^{m-1} i_k = 0,1,2,\ldots,n \right) .
\] (2.13)

From (2.13) we obtain the desired IHDR on \( \mathcal{P} \):
\[
\hat{T}_p = n D(T_p) x_m^n + n \sum_{i=1}^{m} D(T_p) x_m^n \xi_i + \sum_{i=1}^{m} \sum_{k=1}^{m} D(T_p) x_m^n \xi_i \frac{\partial}{\partial x_m} \xi_k + \sum_{k=1}^{m} D(T_p) x_m^n \xi_k \frac{\partial}{\partial x_m} \xi_i .
\] (2.14)
(2.15), e.g., it is also the differential realization of \( L \). Therefore the positive integer \( n \) in (2.14) can be extended to an arbitrary real number.

In comparison to the more difficult situation presented in Appendix A of Ref. 11 with \( su(3) \) as an example, the IHDR obtained in this paper looks very trivial because it only covers those representations that are the symmetrized direct product of one fundamental representation [e.g., a triplet or an antitriplet for \( su(3) \)] and are marked by one positive integer \( n \). It is well known that the finite-dimensional (irreducible) representations of semisimple Lie algebras are marked by rank \( L \) non-negative integers, where rank \( L \) is the rank of semisimple Lie algebra \( L \). In order to obtain all representations of semisimple Lie algebras in the space of polynomials, all the fundamental representations must be exhausted. In the product of all the fundamental representations we impose certain additional conditions and then obtain the nontrivial IHDR marked by rank \( L \), one of which is the number of a fundamental representation in the product. However, it is difficult to obtain analytically such an IHDR in practice. Because the main purpose of this paper is to obtain the indecomposable representations in a differential way, we only need the IHBR that corresponds to the most trivial IHDR (2.14). In the following discussions we will see that the indecomposable representations are marked by a certain number of complex numbers involving \( n \) on the quotient spaces of the universal enveloping algebra of Heisenberg–Weyl algebra.

B. IHBR of Lie algebras

Notice the corresponding relation between creation and annihilation operators of \( (m - 1) \)-boson states \( \{ a^+; \}, a_i = 1,2, ..., m - 1 \) and the operators \( \{ \xi_i, \partial / \partial \xi_i \}; i = 1,2, ..., m - 1 \) in \( \mathcal{P} \),

\[
\{ \xi_i, \partial / \partial \xi_i \} = \delta_{ij} 1, \quad \{ \xi_i, \xi_j \} = \left[ 1, \frac{\partial}{\partial \xi_i} \right] = 0.
\]

We obtain the IHBR of Lie algebras from (2.14):

\[
B(T_p) = nD(T_p)_{mm} + n \sum_{i=1}^{m-1} D(T_p)_{im} a_i^+ + \sum_{k=1}^{m-1} D(T_p)_{mk} a_k^+ + \sum_{i=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{ik} a_i^+ a_k^+ - D(T_p)_{mm} \sum_{k=1}^{m-1} a_k^+ a_k^+ - \sum_{i=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{im} a_i^+ a_k^+ a_k.
\]

When we regard \( E \) as 1 we have

\[
[B(T_p), B(T_q)] = \sum_r C_{pq}^r B(T_r).
\]

Although the IHDR and the IHBR of Lie algebras satisfy the same commutation relations, the IHBR can give richer representations than the IHDR. In fact, \( \mathcal{P} \) is isomorphic to the subspace \( \mathcal{F}(n) \), spanned by

\[
\{ a_i^+; a_i^+; \cdots; a_i^+; a_{m-1}^+ \} \mid i_1, i_2, ..., i_{m-1} \in \mathbb{Z}^+, \\
i_1 + i_2 + \cdots + i_{m-1} = 1,2, ..., n, \quad a_k \{ 0 \} = 0 \},
\]

of the Fock space \( \mathcal{F} \) with basis

\[
\mathcal{F}: \{ a_i^+; a_i^+; \cdots; a_i^+; a_{m-1}^+ \} \mid i_1, i_2, ..., i_{m-1} \in \mathbb{Z}^+, \\
a_k \{ 0 \} = 0 \}.
\]

The Fock space \( \mathcal{F} \) is isomorphic to the quotient space \( \mathcal{F}' \equiv \Omega(\mathcal{H})/\Omega \) of the universal enveloping algebra \( \Omega \) of the \( (m - 1) \)-state Heisenberg–Weyl algebra \( \mathcal{H}: \{ a_i^+, a_i, E \}; i = 1,2, ..., m - 1 \) with the PBW basis

\[
\Omega: \left\{ X(r_1,s_1,t) \equiv \left( \prod_{i=1}^{m-1} a_i^+ \right)^{r_1} \left( \prod_{t=1}^{m-1} a_t^+ \right)^{s_1} \right\} r_1,s_1,t \in \mathbb{Z}^+, \quad \text{I}\right\}
\]

where \( I \) is a left ideal generated by \( (E - 1) \), and \( J \) is a left ideal generated by \( \{ a_i; i = 1,2, ..., m - 1 \} \). The space \( \mathcal{V} \equiv \Omega/ I \) with basis

\[
\mathcal{V}: \{ X(r_1,s_1,t) \equiv \left( X(r_1,s_1,t) \bmod I \right) r_1,s_1,t \in \mathbb{Z}^+ \},
\]

which carries the representation \( p(Q(E) = 1) \) of \( \mathcal{H} \), is larger than \( \mathcal{F}' \). Therefore the representations of \( B(T_p) \) on \( \mathcal{V} \) are richer than the representations of \( \mathcal{T}_p \) on \( \mathcal{P} \). This is why we study the representations by making use of the IHBR, instead of the IHDR.

Comparing with the homogeneous boson realization given in Refs. 5 and 6, the IHBR has merit: It only uses creation and annihilation operators of \( (m - 1) \)-boson states for the \( m \)-dimensional faithful representation \( D \) of Lie algebra \( L \), while the homogeneous boson realization must use creation and annihilation operators of \( m \)-boson states.

C. Example: IHDR and IHBR of \( sl(m) \) algebra

We choose the basis for \( sl(m) \) algebra as

\[
\{ T_{ij}; e_{ij}; (i \neq j) = 1,2, ..., m \}, \\
\{ T_k = e_{kk} - e_{k(k+1)(k+1)} \}; \\
(k = 1,2, ..., m - 1),
\]

(2.22)

where \( e_{ij} \) is a \( m \times m \) matrix with matrix element \( (e_{ij})_{pq} = \delta_{pq} \delta_{ij} \). The finite-dimensional representation \( D \) of \( sl(m) \) algebra is chosen as the natural representation, e.g., \( D(T_{ij}) = D(T_{ik}) D(T_{kj}) = T_{kj} \). By making use of Eq. (2.14) we obtain the IHDR of \( sl(m) \) algebra:

\[
\tilde{T}_{pq} = \xi_p \frac{\partial}{\partial \xi_q} \quad (p \neq q = 1,2, ..., m - 1),
\]

\[
\tilde{T}_{pm} = n \xi_p \sum_{j=1}^{m-1} \xi_j \frac{\partial}{\partial \xi_i} \quad (p = 1,2, ..., m - 1),
\]

\[
\tilde{T}_{mq} = \frac{\partial}{\partial \xi_q} \quad (q = 1,2, ..., m - 1),
\]

\[
\tilde{T}_p = \xi_p \frac{\partial}{\partial \xi_q} - \xi_{p+1} \frac{\partial}{\partial \xi_{p+1}} \quad (p = 1,2, ..., m - 2),
\]

(2.23)
\[ \hat{T}_{m-1} = \xi_{m-1} \frac{\partial}{\partial \xi_{m-1}} - n + \sum_{j=1}^{m-1} \xi_j \frac{\partial}{\partial \xi_j}, \]

The corresponding IHBR is

\[ B(T)p = a^+_p a_q \quad (p \neq q = 1, 2, \ldots, m - 1), \]
\[ B(T)p = n a_p - a^+_p \sum_{j=1}^{m-1} a_j^+ a_j \quad (p = 1, 2, \ldots, m - 1), \]
\[ B(T)q = a_q \quad (q = 1, 2, \ldots, m - 1), \]
\[ B(T)p = a^+_p a_p - a^+_{p+1} a_{p+1} \quad (p = 1, 2, \ldots, m - 2), \]
\[ B(T)m-1 = a^+_m a_{m-1} - n + \sum_{j=1}^{m-1} a_j^+ a_j. \]

When \( m = 2 \) we obtain the IHDR and IHBR of \( sl(2) \) algebra:

\[ \hat{T}^+ = n \xi - \xi \frac{d}{d \xi}, \quad \hat{T}^- = -\frac{1}{2} n + \xi \frac{d}{d \xi}, \]
\[ B(T^+) = n a^+ - a^+ a^+, \quad B(T^-) = a, \]
\[ B(T^0) = -\frac{1}{2} n + a^+ a. \]

The corresponding IHBR is

\[ \rho(a^+_p)X(r,s,t) = X(r, \delta s + \delta t), \]
\[ \rho(a_p)X(r,s,t) = X(r, s + \delta t) \]
\[ + r_k X(r - \delta_k s + \delta t + 1), \]
\[ \rho(E)X(r,s,t) = X(r,s,t) \quad [\text{e.g., } \rho(E) = 1]. \]

By making use of the equation

\[ \Gamma (T_p) = nD(T_p) mm + n \sum_{i=1}^{m-1} D(T_p) jm \rho(a^+_j) + \sum_{k=1}^{m-1} D(T_p) jk \rho(a_k) + \sum_{i=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) jm \rho(a^+_j) \rho(a_k), \]
we obtain the representation \( \Gamma \) of the Lie algebra \( L \) on \( V \):

\[ \Gamma (T_p) X(r, s, t) = \left( n - \sum_{k=1}^{m-1} r_k \right) \sum_{j=1}^{m-1} D(T_p) jm X(r, \delta_j s, \delta_j t) + \left( n - \sum_{k=1}^{m-1} r_k \right) D(T_p) mm X(r, s, t) \]
\[ - \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) jm X(r + \delta_j s, \delta_j t) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) jk X(r, \delta_j s + \delta_j t, \delta_j k) \]
\[ - D(T_p) mm \sum_{k=1}^{m-1} X(r, \delta_k s, \delta_k t) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) zk X(r, \delta_k s + \delta_k t) \]
\[ + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) jk r_k X(r, \delta_k s, \delta_k t) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p) zk r_k X(r, \delta_k s). \]

It is observed that the value \( \sum_{i=1}^{m-1} s_i \) cannot decrease in Eq. (3.4). Thus each non-negative integer \( N \) defines a \( \Gamma \)-invariant subspace \( V^{[N]} \) of \( V \) with basis

\[ V^{[N]}: \left\{ X(r, s) \left| \sum_{i=1}^{m-1} s_i > N, \quad r, s \in \mathbb{Z}^+ \right. \right\}, \]

for which no invariant complementary subspace exists. Thus the representation (3.4) on \( V \) is indecomposable. The representation subduced on every \( V \) is also indecomposable.

It is easy to see that there exists an invariant subspace chain of the space \( V \):

\[ V^{[0]} \supset V^{[1]} \supset V^{[2]} \supset \cdots \supset V^{[N]} \supset \cdots \]

Correspondingly, there are quotient spaces \( V^{[N,K]} = V^{[N]}/V^{[N+K]} \):

\[ V^{[N,K]}: \left\{ Y(r, s, t) = X(r, s) \text{mod } V^{[N+K]} \left| N < \sum_{i=1}^{m-1} s_i < N + K - 1 \right. \right\}, \quad N \in \mathbb{Z}^+, \quad K \in \mathbb{N}. \]

The representation on \( V^{[N]} \) can induce a representation \( \overline{\Gamma} \) on \( V^{[N,K]} \). When \( K > 2 \), the representation \( \overline{\Gamma} \) on \( V^{[N,K]} \) is indecomposable. When \( K = 1 \), the representation \( \overline{\Gamma} \) on \( V^{[N,1]} \) becomes
\[
\Gamma \left( T_p \right) Y(r, s) = \left( n - \sum_{i=1}^{m-1} r_i \right) \sum_{j=1}^{m-1} D(T_p)_{jm} Y(r_i + \delta y s_i) + \left( n - \sum_{k=1}^{m-1} r_k \right) D(T_p)_{mm} Y(r, s) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{jk} r_k Y(r_j - \delta k + \delta y s_j) + \sum_{k=1}^{m-1} D(T_p)_{mk} r_k Y(r_i - \delta k, s_i) \left( \sum_{i=1}^{m-1} s_i = N \right). 
\]

(3.8)

It is noted that the values \( s_i \) \((i = 1, 2, \ldots, m - 1) \) do not change. Thus every set \((s_1, s_2, \ldots, s_{m-1})\) that satisfies the condition \( \sum_{i=1}^{m-1} s_i = N \) defines a \( p \)-invariant subspace \( V^{[N]} \) with basis

\[
V^{[N]} = \left\{ Y(r, s) \in V^{[N]} \left| \sum_{i=1}^{m-1} s_i = N, s_1, s_2, \ldots, s_{m-1} \text{ are fixed} \right. \right\}. 
\]

(3.9)

The representation on \( V^{[N]} \) is completely reducible:

\[
V^{[N]} = \bigoplus_{(s_1, \ldots, s_{m-1}) \subseteq \{0, 1, \ldots, N\}} V^{[N]} (s_1, \ldots, s_{m-1}) . 
\]

(3.10)

When \( n \) is not a non-negative integer, the representation subduced on every \( V^{[N]} (s_1, \ldots, s_{m-1}) \) is irreducible. When \( n \in \mathbb{Z}^+ \), it is obvious that there exists an invariant subspace \( V^{[N]} (s_1, \ldots, s_{m-1}) (n) \) of \( V^{[N]} (s_1, \ldots, s_{m-1}) \):

\[
V^{[N]} (s_1, \ldots, s_{m-1}) (n) = \left\{ Y(r, s) \in V^{[N]} (s_1, \ldots, s_{m-1}) \left| \sum_{i=1}^{m-1} r_i \leq n, \ r_i \in \mathbb{Z}^+ \right. \right\}, 
\]

(3.11)

with the dimension

\[
\dim V^{[N]} (s_1, \ldots, s_{m-1}) (n) = \sum_{k=1}^{n} \frac{(m + k - 2)!}{k! (m - 2)!}, 
\]

(3.12)

for which no invariant complementary subspace exists. Thus the representation subduced on \( V^{[N]} (s_1, \ldots, s_{m-1}) \) is indecomposable. The subspace \( V^{[N]} (s_1, \ldots, s_{m-1}) (n) \) carries a finite-dimensional irreducible representation.

The relation \( \{a_i - \Lambda_i \mid a_i \in \mathbb{C}, i = 1, \ldots, m - 1\} \) generates a left ideal \( J' \) of \( V \). For the quotient space \( V' = V / J' \), also called the Fock space, a basis can be chosen as

\[
(3.13)
\]

The representation (3.4) on \( V \) induces a representation on \( V' \),

\[
\Gamma \left( T_p \right) X(r) = \sum_{j=1}^{m-1} \left[ \left( n - \sum_{i=1}^{m-1} r_i \right) D(T_p)_{jm} + \sum_{k=1}^{m-1} D(T_p)_{jk} \Lambda_k - D(T_p)_{mm} \Lambda_r \right] X(r_i + \delta y) 
- \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{jm} \Lambda_k X(r_i + \delta k + \delta y) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{jk} r_k X(r_i - \delta k + \delta y) 
+ \sum_{k=1}^{m-1} D(T_p)_{mk} r_k X(r_i - \delta k) + \left( n - \sum_{i=1}^{m-1} r_i \right) D(T_p)_{mm} + \sum_{k=1}^{m-1} D(T_p)_{mk} \Lambda_r \right] X(r_i), 
\]

(3.14)

which is an infinite-dimensional irreducible representation for the case with \( \Lambda_i \neq 0 \). When \( \Lambda_1 = \Lambda_2 = \cdots = \Lambda_{m-1} = 0 \), Eq. (3.14) becomes

\[
\Gamma \left( T_p \right) X(r) = \left( n - \sum_{i=1}^{m-1} r_i \right) \sum_{j=1}^{m-1} D(T_p)_{jm} X(r_i + \delta y) + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} D(T_p)_{jk} r_k X(r_i - \delta k + \delta y) 
+ \sum_{k=1}^{m-1} D(T_p)_{mk} r_k X(r_i - \delta k) + D(T_p)_{mm} \left( n - \sum_{i=1}^{m-1} r_i \right) X(r_i). 
\]

(3.15)

The representation (3.15) is equivalent to the representation on every \( V^{[N]} (s_1, \ldots, s_{m-1}) \). When \( n \in \mathbb{Z}^+ \), (3.15) is an infinite-dimensional representation. When \( n \in \mathbb{Z}^+ \), (3.15) is an indecomposable representation in which there exists a finite-dimensional irreducible representation on the invariant subspace of \( V' \), \( \mathcal{F}' \),

\[
\mathcal{F}'(n) = \left\{ X(r_i) \in \mathcal{F}' \left| \sum_{i=1}^{m-1} r_i \leq n, \ r_i \in \mathbb{Z}^+ \right. \right\}, 
\]

(3.16)

with dimension

\[
\dim \mathcal{F}'(n) = \sum_{k=0}^{n} \frac{(m + k - 2)!}{k! (m - 2)!}. 
\]

(3.17)
From the above discussion, we see that we can obtain the finite-dimensional representations on the subspace of Fock space if we use the IHBR of Lie algebra. If we adopt the homogeneous boson realization of Lie algebra, we can only obtain the finite-dimensional representations on the quotient spaces of Fock space (see Refs. 5 and 6).

IV. IHBR OF INDECOMPOSABLE REPRESENTATIONS OF SU(2) ALGEBRA

The representation of the one-state Heisenberg–Weyl algebra $\mathcal{H}$: $\{a^+, a, E\}$ on its universal enveloping algebra $\Omega$ with PBW basis

$$\Omega: \{X(r,s,t) | r,s,t \in Z^+\}$$

is defined as

$$\rho(a^+)X(r,s,t) = X(r+1,s,t),$$

$$\rho(a)X(r,s,t) = X(r,s+1,t) + rX(r-1,s,t+1),$$

$$\rho(E)X(r,s,t) = X(r,s,t+1).$$

The relation $(E-1)$ generates a left ideal $I$ of $\Omega$. The representation (4.2) induces on the quotient space $V = \Omega/I$ with basis

$$V: \{X(r,s) \equiv X(r,s,0) \mod I | r,s \in Z^+\}$$

(4.3)
a representation

$$\rho(a^+)X(r,s) = X(r+1,s),$$

$$\rho(a)X(r,s) = X(r,s+1) + rX(r-1,s),$$

$$\rho(E) = 1.$$ 

By making use of the IHBR (2.25b) of $su(2)$ algebra and the equation

$$\Gamma(T^+ = np(a^+) - [\rho(a^+)]^2 \rho(a), \quad \Gamma(T^-) = \rho(a), \quad \Gamma(T^0) = -n/2 + \rho(a^+) \rho(a),$$

(4.4)

we obtain the representation of $su(2)$ algebra on $V$,

$$\Gamma(T^+)X(r,s) = (n-r)X(r+1,s) - X(r+2,s+1),$$

$$\Gamma(T^-)X(r,s) = X(r,s+1) + rX(r-1,s),$$

$$\Gamma(T^0)X(r,s) = (r-n/2)X(r,s) + X(r+1,s+1).$$

(4.5)

It is easy to see that a non-negative integer $S$ defines an invariant subspace $V^{[S]}$ of $V$,

$$V^{[S]} = \{X(r,s) | s \geq S, \ r,s \in Z^+\},$$

(4.6)

for which no invariant complementary subspace exists. Thus the representation (4.6) is indecomposable. The representation subduced on every $V^{[S]}$ is also indecomposable.

On the invariant subspace chain of $V$,

$$V \equiv V^{[0]} \supset V^{[1]} \supset V^{[2]} \supset \cdots \supset V^{[S]} \supset \cdots,$$

(4.7)

there exist some quotient spaces $V^{[S,K]} = V^{[S]}/V^{[S+K]}$:

$$V^{[S,K]}: \{Y(r,s) \equiv X(r,s) \mod V^{[S+K]} | S \leq S < S + K - 1,$$

$$r,s \in Z^+, \ K \in N.$$ (4.8)

When $K > 1$, the representation induced on $V^{[S,K]}$ is indecomposable. When $K = 1$, the representation induced on $V^{[S+1]}$ is

$$\Gamma(T^+)Y(r,s) = (n-r)Y(r+1,s),$$

$$\Gamma(T^-)Y(r,s) = rY(r-1,s),$$

$$\Gamma(T^0)Y(r,s) = (r-n/2)Y(r,s).$$

(4.9)

Equations (4.10) constitute an infinite-dimensional irreducible representation for the case with $n \in Z^+$. If $n \in Z^+$, there exists an $(n+1)$-dimensional subspace $V^{[S+1]}(n)$ of $V^{[S]}$ with basis

$$V^{[S,1]}: \{Y(r,s) \in V^{[S]} | r < n, \ r \in Z^+\}$$

(4.11)

for which no invariant complementary subspace exists. Thus Eqs. (4.10) are an indecomposable representation for the case with $n \in Z^+$. If we define a new basis for $V^{[S,1]}(n)$,

$$|j,m\rangle = (1/\sqrt{(j+m)! (j-m)!})Y(j+m,s),$$

(4.12)

where $j = n/2$, $m = -j, -j + 1, \ldots$, the representation subduced on $V^{[S,1]}(n)$ becomes

$$\Gamma(T^+)|j,m\rangle = \sqrt{(j+m)! (j+m+1)!} |j,m\rangle,$$

$$\Gamma(T^-)|j,m\rangle = \sqrt{(j+m)! (j+m+1)!} |j,m\rangle,$$

$$\Gamma(T^0)|j,m\rangle = m |j,m\rangle,$$

(4.13)

which is an irreducible representation of $su(2)$ of the highest weight $j$ with dimension $(2j+1)$.

The relation $\{a - \Lambda | \Lambda \in C\}$ generates a left ideal $J'$. The representation (4.6) induces on the Fock space $\mathcal{F}' \equiv V/J'$,

$$\mathcal{F}': \{X(r) \equiv X(r,0) \mod J' | r \in Z^+\},$$

(4.14)

a representation

$$\Gamma(T^+)X(r) = (n-r)X(r+1) - \Lambda X(r+2),$$

$$\Gamma(T^-)X(r) = \Lambda X(r) + rX(r-1),$$

$$\Gamma(T^0)X(r) = (r-n/2)X(r) + \Lambda X(r+1).$$

(4.15)

Equations (4.15) are an infinite-dimensional irreducible representation for the case with $\Lambda \neq 0$. If $\Lambda = 0$, (4.15) become

$$\Gamma(T^+)X(r) = (n-r)X(r+1),$$

$$\Gamma(T^-)X(r) = rX(r-1),$$

$$\Gamma(T^0)X(r) = (r-n/2)X(r),$$

(4.16)

which is equivalent to the representation on $V^{[S,1]}$. Equation (4.16) is an infinite-dimensional irreducible representation for the case with $n \in Z^+$. When $n \in Z^+$, there exists an invariant subspace $\mathcal{F}'(n)$ of $\mathcal{F}'$ with dimension $(n+1)$,

$$\mathcal{F}'(n): \{X(r) \in \mathcal{F}' | r < n, \ r \in Z^+\},$$

(4.17)

for which no invariant complementary subspace exists. Thus the representation (4.16) is indecomposable for the case with $n \in Z^+$. If we define a new basis for $\mathcal{F}'(n)$,

$$|j,m\rangle = (1/\sqrt{(j+m)! (j-m)!})Y(j+m,s),$$

(4.18)

where $j = n/2$, $m = -j, -j + 1, \ldots$, the representation subduced on $\mathcal{F}'(n)$ becomes

$$\Gamma(T^+) |j,m\rangle = \sqrt{(j+m)! (j+m+1)!} |j,m\rangle,$$

$$\Gamma(T^-) |j,m\rangle = \sqrt{(j+m)! (j+m+1)!} |j,m\rangle,$$

$$\Gamma(T^0) |j,m\rangle = m |j,m\rangle,$$

(4.19)
which is an irreducible representation of $su(2)$ of the highest weight $j$ with dimension $(2j + 1)$.

For example, when $j = \frac{1}{2}$ and the order of the basis for $Y'$ is chosen as

$$\{|\frac{1}{2}, 0\}, |\frac{1}{2}, -\frac{1}{2}\}, X(2), X(3), X(4), \ldots \}$$

in which $|\frac{1}{2}, 0\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$ are the basis for $Y'(n)$, the representation on $Y'(A = 0)$ is

$$\Gamma(T^+) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & \cdots \\ 0 & -2 & 0 & \cdots \end{pmatrix}$$

$$\Gamma(T^-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & \cdots \\ 0 & 0 & 4 & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix}$$

which is an infinite-dimensional indecomposable representation.

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