The accelerating cosmic expansion is first inferred from the observations of distant type Ia supernovae [1]. It indicates unexpected gravitational physics attributed to the dominating presence of a dark energy with negative pressure. Some other independent observations, such as the cosmic microwave background radiation (CMBR) and Sloan Digital Sky Survey (SDSS), also strongly favor dark energy as the dominant component in the present mass-energy budget of the Universe.

A simple candidate for the dark energy is Einstein’s famous cosmological constant \( \rho_\Lambda \). Nowadays it is still consistent with all of the observations. (See the recent analysis of the experiments in [2,3].) If we take the Planck scale as a natural cutoff for the quantum field theories, the zero point energy density \( \langle \rho \rangle = M_p^4/(16\pi^2) \) is much greater than the observed value of the energy density of dark energy \( \rho_D = 10^{-123}M_p^4 \). The energy scale for the local effective field theory related to the cosmological constant is roughly \( 10^{-3} \text{ eV} \). The puzzle is why the vacuum energy is so small after including all of the zero point energies. Another problem is why the energy density of dark energy is comparable to the matter energy density now (cosmic coincidence problem). For a classic review see [4], for a recent nice review see [5], and for a recent discussion see [6].

Another source for an appropriate dark energy component is the energy density of a single slow-rolling scalar field called quintessence [7]. In an expanding universe, a spatially homogeneous canonical scalar field minimally coupled to gravity with potential \( V(\phi) \) obeys

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0, \tag{1}
\]

where the dot and prime denote the derivative with respect to the cosmic time and quintessence \( \phi \), respectively. The energy density and the pressure of quintessence are respectively \( \rho_Q = \frac{1}{2} \dot{\phi}^2 + V(\phi) \) and \( p_Q = \frac{1}{2} \dot{\phi}^2 - V(\phi) \), and then the equation of state parameter is given by

\[
\frac{w}{p} = \frac{1}{3} \frac{\dot{\phi}^2 - V(\phi)}{\dot{\phi}^2 + V(\phi)}, \tag{2}
\]

which generally varies with time. The range for the equation of state parameter for the quintessence is \( w \in [-1, 1] \). The term \( 3H \dot{\phi} \) acts as a friction term. When the friction term is large enough, the field slowly rolls down its potential and \( \dot{\phi}^2 \ll V(\phi) \). Now \( w \approx -1 \) and the quintessence acts like a cosmological constant.

In the last ten years many quintessence models have been proposed and the equation of state parameter \( w \) evaluated in a wide-range parameter space. In this paper we suggest a possible theoretical constraint on this parameter.

One may expect the quintessence field couples to standard model fields in the absence of forbidding symmetries. Such couplings are normally suppressed by Planck scale. When the scalar field approaches Planck scale, these couplings are no longer suppressed, thereby ruining the “darkness” of the dark energy [8].

Here it is also worth mentioning the case of axion with potential \( V(\phi) = M^4(1 + \cos(\phi)) \) with \( M < M_p \) and \( f < M_p \). In this case the scalar field \( \phi \) does not suffer from a bound to its vacuum expectation value (VEV). In this paper we concentrate on the quintessence models in which the VEV is required not to exceed \( M_p \).

The quintessence model relies on the application of low-energy effective field theory to the quintessence. Here we focus on the quintessence model in which the variation of quintessence is less than \( M_p \). It implies a constraint on the equation of state parameter for the quintessence.

For simplicity of calculations we assume spatial flatness which is motivated by theoretical considerations, such as inflation, and observations. Our results can be easily generalized to the case with a spatial curvature. The Hubble parameter is given by

\[
H^2 = \frac{\rho_{\text{crit}}}{3M_p^2} = \frac{\rho_Q + \rho_m}{3M_p^2}, \tag{3}
\]

where \( \rho_m(z) = \rho_m^0(1 + z)^3 \) is the energy density of the dustlike matter in our universe and \( \rho_m^0 \) is its energy density.
Combining with Eq. (2), we solve (9) as

\[ V(\phi) = \frac{\dot{\phi}^2}{2} \left( 1 - \frac{w}{1 + w} \right). \]

The energy density takes the form

\[ \rho_Q = \frac{1}{2} \dot{\phi}^2 + V(\phi) = \frac{\dot{\phi}^2}{1 + w}. \]

In the whole paper, we assume, without loss of generality, \( V' < 0 \), so that \( \dot{\phi} > 0 \). Thus Eq. (5) reads

\[ \dot{\phi} = \sqrt{(1 + w) \rho_Q}. \]

In fact, tracker is a particularly interesting case which has been studied in [9,10]. Here we consider a more general case. Integrating Eq. (6), we obtain

\[ \frac{|\Delta \phi(z)|}{M_p} = \int_{\phi(0)}^{\phi(z)} \frac{d\phi}{M_p} = \int_0^z \sqrt{3[1 + w(z')]\Omega_Q(z')} \frac{dz'}{1 + z'}. \]

which should be less than 1. Here we use \( H dt = -\frac{dz}{1 + z} \).

The density parameter for the quintessence is

\[ \Omega_Q = \frac{\rho_Q}{\rho_{crit}} = \frac{\rho_Q}{\rho_Q + \rho_m}. \]

The energy conservation implies

\[ \dot{\rho}_Q + 3H(\rho_Q + p_Q) = 0. \]

Combing with Eq. (2), we solve (9) as

\[ \rho_Q(z) = \rho_Q^0 \exp \left( \int_0^z 3 \frac{1 + w(z')}{{1 + z'}} \frac{dz'}{1 + z'} \right). \]

where \( \rho_Q^0 \) is the present energy density of quintessence. Using (10), we obtain

\[ \frac{1}{\Omega_Q(z)} = 1 + \frac{\Omega_m}{\Omega_Q} (1 + z)^3 \exp \left( -\int_0^z 3 \frac{1 + w(z')}{{1 + z'}} \frac{dz'}{1 + z'} \right). \]

For the case with a spatial curvature, we only need to add another term \( -\frac{\Omega_k}{\Omega_Q} (1 + z)^3 \exp \left( -\int_0^z 3 \frac{1 + w(z')}{{1 + z'}} \frac{dz'}{1 + z'} \right) \) on the right-hand side of Eq. (11). Here we set \( \Omega_k^0 = 0 \).

We will use the condition \( |\Delta \phi(z)|/M_p < 1 \) to constrain the equation of state parameter \( w(z) \) for the quintessence. Unfortunately, present dynamical dark energy models in the literature do not suggest a universal or fundamental parametric form for \( w(z) \). (For a recent review see [11].)

We will investigate several typical parametrizations of \( w(z) \). There are also strong degeneracies in the effect of \( w(z) \) and \( \Omega_m \) on the expansion history. According to the literature, we reasonably set \( \Omega_Q^0 = 0.72 \) and \( \Omega_m^0 = 0.28 \).

I. \( w = w_0 = \text{const} \)

There are a few models of quintessence which imply a constant equation of state parameter, but different from the cosmological constant \( (w = -1) \). In this case we consider the variation of the quintessence from now to the last scattering \((z_{rec} = 1089)\). Requiring \( |\Delta \phi(z_{rec})|/M_p < 1 \) yields \( w = w_0 \leq -0.738 \).

In a spatially flat universe, the combination of WMAP and the Supernova Legacy Survey (SNLS) data yields a significant constraint on the equation of state parameter for the dark energy \( w = -0.967^{+0.071}_{-0.072} \) [3]. The theoretic limit on the equation of state parameter for the quintessence is consistent with the experiments.

II. \( w = w_0 + w_1 z \)

In this case the equation of state parameter is a linear function of the redshift \( z \). This parametrization is studied in [12]. It is a good parametrization at a low redshift. But in this form, \( w(z) \) diverges, making it unsuitable at high redshift. As we know, the redshift of the SN sample is less than 2. For the consistency we require \( |\Delta \phi(z = 2)|/M_p < 1 \). The theoretic constraints are \(-1 \leq w_0 \leq -0.164 \) and \(-0.417 \leq w_1 \leq 0.854 \). Here we also consider the requirement \( w \in [-1, 1] \) for the quintessence. A more explicit result is showed in Fig. 1.

III. \( w = w_0 + w_1 \frac{z}{1 + z} \)

This parametric form is suggested in [13]. It solves the divergence problem in case II and has been widely used in

![FIG. 1.](103518-2.png)

The gray patch is the prediction of the quintessence. The light gray patch corresponds to \( w < -1 \). The line with \( |\Delta \phi(z = 2)|/M_p = 1 \) is roughly a straight line which is linearly fitted as \( w_0 + 0.657w_1 = -0.456 \).
The light gray patch corresponds to the prediction of the quintessence. The authors in [15] proposed that the equation of state parameter for phantom is out of the question. On the other hand, NEC is a crucial assumption in proving the positivity of the ADM mass in asymptotically flat space [19]. The positive energy theorem implies a stable vacuum for gravity and will play a crucial role in quantum gravity.

We would like to thank F. L. Lin and P. J. Yi for useful discussions.

