A GUIDED TOUR TO APPLIED SYMBOLIC DYNAMICS

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Abstract

A brief introduction to recent development in applied symbolic dynamics with a list of references is given.

1 Abstract versus Applied Symbolic Dynamics

For physicists symbolic dynamics is nothing but coarse-grained description of dynamics. The idea dated back to the work of J. Hadamard[1] and M. Morse[2] at the turn of 20th century.

Abstract formulation of symbolic dynamics goes as follows. Let $f: \ M \to M$ be a diffeomorphism of a compact manifold M to itself. Since M is compact, one may choose a finite covering and label this covering by using letters (symbols) from a finite alphabet $\mathcal{A}$. Then any orbit $\{f^n(x)\}_{n=0}^{\infty}$ (for simplicity we take semi-infinite orbits only) corresponds to a symbolic sequence $s_0 s_1 s_2 s_3 \cdots$, which is considered as a point in the space $\mathcal{S}$ of all possible symbolic sequences made of letters from the alphabet $\mathcal{A}$. One iteration of f corresponds to a shift in the sequence. Thus dynamics on M corresponds to shift automorphism of S. This correspondence has provided a powerful tool for theorem-proving, say, in ergodic theory. For more information one may consult [3] and references therein.

However, one cannot get very far in such a general setting. In order to gain more detailed knowledge on the dynamics, one has to specify M and f, for example, taking M to be an one-dimensional interval or two-dimensional plane and f to be nonlinear functions of a certain class. Since dissipation reduces effectively the dimension of the phase space, this approach works well for dissipative systems. This is what we call *Applied Symbolic Dynamics*.

In a sense applied symbolic dynamics originated from the 1973 paper of Metropolis, Stein and Stein[4] on unimodal maps of the interval. Futher development may be found in [5, 6]. For reviews of recent results see [7, 8, 9].

2 One-Dimensional versus Two-Dimensional Symbolic Dynamics

Since recent development of applied symbolic dynamics of one-dimensional maps has been summarized on another occasion[10] in Korea, we confine ourselves to the essence of symbolic dynamics in two dimensions.

The success of symbolic dynamics of one-dimensional maps is largely based on the nice ordering property of real numbers, which no longer exists in a plane. Moreover, since two-dimensional maps are in general invertible, one now deals with bi-infinite symbolic sequences like

$$\cdots s_{n-2}s_{n-1} \bullet s_n s_{n+1} \cdots,$$

where $\bullet$ denotes the present time: $\bullet s_n s_{n+1} \cdots$ is called the forward symbolic sequence and $\cdots s_{n-2}s_{n-1}\bullet$ — the backward symbolic sequence.

One way of overcoming the ordering difficulty consists in decomposition of the plane into two families of one-dimensional curves, which intersect each other transversely. Then one can order one family of curves along another, still making use of the ordering property in one dimensional. If one can decompose the plane in such a way that all points which lead to one and the same forward symbolic sequences would make one curve in the first family, while all points possessing one and the same backward symbolic sequences would be located along a curve of the second family, the core of the problem would be solved.

The above requirement may be met by using so-called dynamical foliations of the phase plane[11, 12]. Furthermore, some curves from one family of foliations may touch curves of the other family tangentially, not transversely. However, this is not a drawback. Experience of one-dimensional symbolic dynamics tells us that now one has to introduce another symbol with a different "monotonicity". In other words, the tangency points of the two family of foliations determine a partition of the phase plane.

Grassberger and Kantz[13] first suggested to determine partition lines by tangencies between the invariant stable and unstable manifolds of fixed or periodic points. These manifolds are subsets of the dynamical foliations mentioned above. This is enough to study the dynamics within the attractor. When the partition of the whole phase plane is needed, one has to generalize the tangencies to that between the two dynamical foliations[19, 20, 24]. The symbolic dynamics of the Tél map[14], the Lozi map[15], the Hénon map[16], and some other maps has been studied in detail[17]–[26].

3 Applications to Ordinary Differential Equiations

An important application of symbolic dynamics of low-dimensional maps consists in the study of Poincaré sections of ordinary diffential equations. In the 1980s, the forced Brusselator and the Lorenz model were studied by using symbolic dynamics of one-dimensional maps[27, 28]. These studies have raised many questions which can be answered only by invoking symbolic dynamics of two-dimensional maps. We could return to these questions only when the latter has been well understood. Recently, our group has made significant progress in the study of the pendulum equations, the forced Brusselator, the Duffing equations, the extended Bloch model for NMR-laser, and the Lorenz equations[29]–[36]. This combined use of numerical work with the method of symbolic dynamics, which is topological in nature, looks quite promising. It also calls for a higher degree of automation, which may be accomplished when we will have accummulated more experience in dealing with various systems, periodically forced as well as autonomous.

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