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Number of Periodic Orbits in Continuous Maps of the Interval—Complete Solution of the Counting Problem

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Abstract. The problem of counting the number of types of periodic orbits in continuous maps of the interval has been solved completely by using several different methods. We summarize the results without going into details which have been published elsewhere [3–5].

Keywords: mappings of interval, periodic orbits, symbolic dynamics, necklace problem

1. Introduction

In non-linear dynamical systems periodic orbits, stable as well as unstable, have a close relation with chaotic behavior. On the one hand, when a parameter is tuned, chaotic regime is usually reached via a sequence of stable periodic events, the period-doubling cascade being a prominent example. On the other, a chaotic attractor that appears at a certain parameter contains an infinite number of unstable periodic orbits. The type and number of periodic orbits are topological invariants and thus may be used to characterize the attractors. Moreover, higher-dimensional *dissipative* systems may have the same kind of attractors as low-dimensional ones. In particular, the number of periodic orbits in one-dimensional maps may be instructive for the study of higher-dimensional systems.

The number of periodic orbits in unimodal maps has been known for some time [6]. There has been some confusion and clarification on the number of periods in cubic maps [4, 7, 9, 13, 14]. Counting the number of periodic orbits in maps with multiple critical points seemed to be a quite difficult job until the realization that one has first to solve the problem for a family of one-parameter maps and the answer to more complicated cases is given by combinations of results of the one-parameter cases.

2. Statement of the Counting Problem

We consider a general family of *continuous* maps of the interval I:

$$f_\mu : I \mapsto I,$$

or

$$x_{n+1} = f(\mu, x_n), \quad x \in I,$$

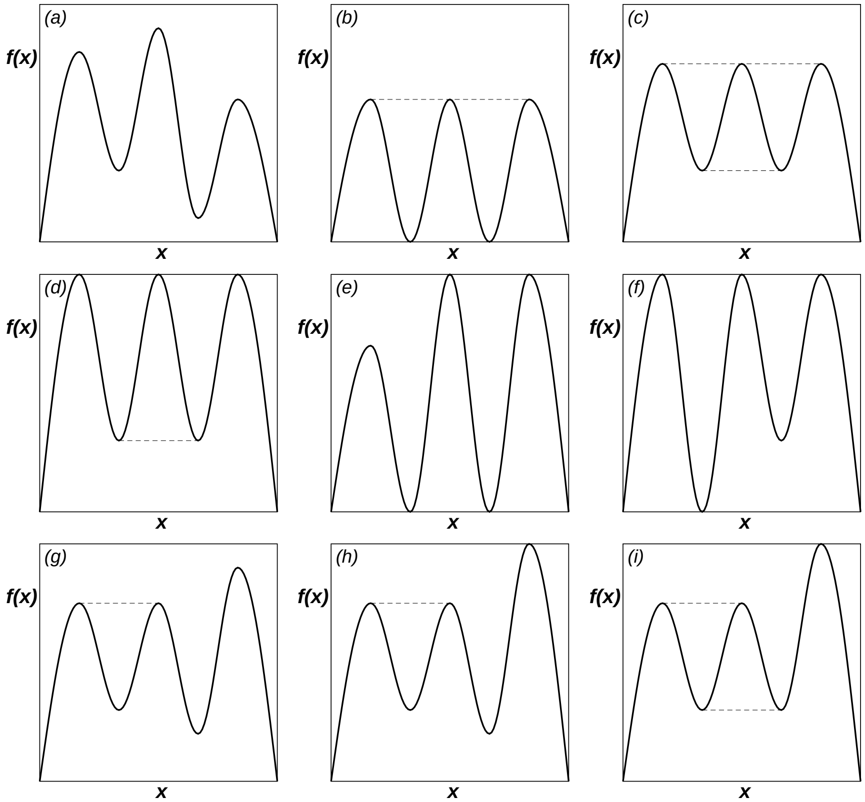

Figure 1: Particular cases of an $m = 6$ map.

where μ denotes one or more parameters. The mapping function may have m monotone branches or *laps*, separated by *turning points*, i.e., maxima and minima of the function. Without loss of generality we can take the interval to be $I = (0, 1)$. In order to obtain general results, one has to fix the function f at the two end points of the interval:

$$f(0) = 0,$$

$$f(1) = \begin{cases} 0, & m \quad \text{even}, \\ 1, & m \quad \text{odd}. \end{cases}$$

The counting problem consists in telling the number of different types of periodic orbits of a given length n when the parameters are allowed to change in all possible ways that keep the map within the unit square.

3. Summary of the Results

An m-lap continuous map has $m-1$ critical points, among which there are $[m/2]$ maxima and $[(m-1)/2]$ minima, where $[\,]$ means taking the integer part. Some of these minima or maxima or both may be bound to vary in unison, thus reducing the number of free parameters. Figure 1 shows some particular cases of an $m=6$ map. Figure 1(b) shows a one-parameter case when all the minima are fixed at the lowest point $f(x)=0$ while all the maxima are bound to vary together. We denote the number of period n orbits in this one-parameter map by $N_6(n)$. If m is odd, a one-parameter map is obtained by fixing the rightmost point at $f(x)=1$. The one-parameter family of maps is shown in Figure 2 for $m=2$, 3, and 4. In general, the number of period n orbits in such one-parameter maps is denoted by $N_m(n)$. The first $N_m(n)$ are listed in Table 1 for $m=2$ to 7 and $n \le 10$.

It turns out that the number of periodic orbits in all other cases are given by linear combinations of $N_k(n)$ with $k \le m$. Using various cases of the $m=6$ map shown in Figure 1 as examples, we list some results:

1. The number of periods of the one-parameter map (e) (as well as (f) and other similar cases) is given by $N_6(n)-N_4(n)$. In an m-lap map, it is given by $N_m(n)-N_{m-2}(n)$.
2. The number of periods of the map (d) is given by $N_6(n)-N_2(n)$. A similar case in general would have $N_m(n)-N_{m-2\times 2}(n)$ periods.
3. The number of periods of the map (i) is twice that of (d), i.e., $2(N_6(n)-N_2(n))$.
4. The number of periods of the map (c) is the sum of (b) and (d), i.e., $2N_6(n)-N_2(n)$.
5. The number of periods of the map (h) is the sum of (d) and twice of (e), i.e., $3(N_6(n)-N_2(n)-2N_4(n))$.
6. The number of periods of the map (g) is the sum of (d) and thrice that of (e), i.e., $4(N_6(n)-N_2(n)-3N_4(n))$.
7. Finally, the number of periods in the general $m=6$ map (a) is $5(N_6(n)-N_4(n))$. It is $(m-1)(N_m(n)-N_{m-2}(n))$ for a general m-lap map.

If an m-lap map has k_1 critical points which change independently, k_2 pairs of maximal (or minimal) points which change simultaneously, k_3 triples of maximal (or minimal) points which change simultaneously, and so on, then the total number of period n

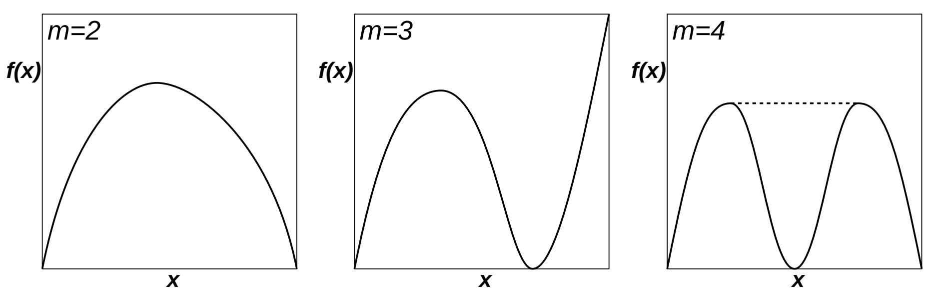

Figure 2: The family of one-parameter maps for $m=2$, 3, and 4.

Table 1: Number of period n orbits $N_m(n)$ for maps with m laps.

$n \setminus m$	2	3	4	5	6	7
1	1	1	2	2	3	3
2	1	2	4	6	9	12
3	1	4	10	20	35	56
4	2	10	32	78	162	300
5	3	24	102	312	777	1680
6	5	60	340	1300	3885	9800
7	9	156	1170	5580	19995	58824
8	16	410	4096	24414	104976	360300
9	28	1092	14560	108500	559860	2241848
10	51	2952	52428	488280	3023307	14123760

superstable orbits is given by

$$N = \sum_{i=1} k_i [N_m(n) - N_{m-2i}(n)],$$

where

$$k_0 + k_1 \times 1 + k_2 \times 2 + k_3 \times 3 + \cdots = m - 1,$$

k_0 being the number of extremes fixed at the top or bottom of the unit square.

The key numbers $N_m(n)$ may be calculated by several different methods. We list a few.

3.1. The Number of Admissible Words in Symbolic Dynamics

One can construct the symbolic dynamics of the map and formulate the admissible conditions for words that are allowed in the dynamics (for details see Chapter 3 of [5]). This method also works for maps with discontinuities. A general program to perform the job is given in Appendix A of [5]. In fact, all the results given below have been checked against this brute force approach.

3.2. The Necklace Problem

The numbers $N_2(n)$ are known to be given by the number of necklaces made of n beads that come in two colors[6], i.e., the number of periodic sequences that are invariant under the group $C_n \times S_2$, where C_n is the cyclic group of order n and S_2 the symmetric group of order 2. There was a misconnection of the group $C_n \times S_3$ to $N_3(n)$ [9]. It turns out that the general case is still given by the group $C_n \times S_2$, but it is no longer a necklace problem. The corresponding counting formula reads:

$$F_m^*(n) = \frac{1}{2n} \sum_{d|n} \varphi(d) (m^{\frac{n}{d}} + \overline{m}^{\frac{n}{d}}),$$

Table 2: Value of $\overline{m}$.

	m even	m odd
d even	m	m
d odd	0	1

where the sum runs over all factor d of n and $\varphi(d)$ is the Euler function. The number $\overline{m}$ is defined according to the parity of m and d as listed in Table 2.*

The numbers $F_m^*(n)$ contain those n which are multiples of a shorter period. The final result for $N_m(n)$ is obtained from F_m^* by a Möbius transform:

$$N_m(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F_m^*(d),$$

where $\mu(n)$ is the Möbius function. We note that the above counting formula looks much simpler than the corresponding formula for the number of periodic sequences invariant under the group $C_n \times S_2$, given by Gilbert and Riordan [2].

3.3. Recursion Formulae

Let us denote by $P_m(n)$ the number of period n orbits that are born in period-doubling, i.e., pitch-fork, bifurcations, and by $M_m(n)$ the number of period n orbits born at tangent, i.e., saddle-node, bifurcations. We have by definition

$$\begin{aligned} P_m(2k+1) &= 0, & \text{for} \quad k \geq 0, \\ P_m(2k) &= P_m(k) + M_m(k), & \text{for} \quad k \geq 1. \end{aligned}$$

Furthermore, let

$$C_m(n) \equiv d[2M_m(d) + P_m(d)].$$

Then the numbers $C_m(n)$ are given by a "balance equation"

$$m^n = \sum_{d|n} C_m(d) + s(m),$$

where

$$s(m) = m \pmod 2.$$

The above relations provide a recursion scheme for the calculation of $P_m(n)$ and $M_m(n)$. One can solve the last relation by using the Möbius transform to yield

$$C_m(n) = \sum_{d|n} \mu(d) m^{\frac{n}{d}} - s(m) I(n),$$

where $I(n) = [1/n]$. We see that the factor $s(m)$ affects only the number of fixed points when m is odd. This is solely due to the fact that for odd m, the rightmost point $f(1) = 1$ is always a fixed point.

* We take this opportunity to correct a typesetting mistake in Table III of [10] and in Table 7.4 of [5].

The total number of periods $N_m(n) = P_m(n) + M_m(n)$ is given by a similar recursion formula:

$$N_m(n) = \frac{1}{2n} \sum_{d|n'} \mu(d) m^{\frac{n}{d}} - s(m) I(n'),$$

where n' is obtained from n by decomposing the latter as

$$n = 2^k n', \quad k \geq 0, \quad n' \text{ odd}.$$

3.4. Finite λ Auto-Expansion of Real Numbers

Take an m-lap piecewise linear function that maps the interval $(0,\ m)$ into itself:

$$f(x) = \lambda\alpha_i x - \lambda\beta_i, \quad \lambda \in (1, m),$$

where

$$\begin{aligned} &\alpha_i = 1, \quad \beta_i = 2i, \quad \text{for } 2i \leq x \leq 2i+1,\ i = 0, 1, \ldots, [(m-1)/2]; \\ &\alpha_i = -1,\ \beta_i = -2i,\ \text{for } 2i-1 \leq x \leq 2i,\ i = 1, 2, \ldots, [m/2]; \end{aligned}$$

and define

$$\begin{aligned} A_i &= 2i\alpha_{i-1} \cdots \alpha_0, \quad i = 0, 1, \ldots, n-2, \\ A_{n-1} &= [2(n-1) + \alpha_{n-1}]\alpha_{n-2} \ldots \alpha_1 \alpha_0, \end{aligned}$$

we ask how many $\lambda \in (1,\ m)$ would have a finite expansion

$$\lambda = \sum_{i=0}^{n-1} \frac{A_i}{\lambda^i},$$

which is a natural extension of binary expansion of real numbers. It turns out that the number is also connected with the number of periodic orbits in the one parameter m-lap map. This is a straightforward extension of the result by Derrida et al. [1] for $m = 2$.

We did not touch some other aspects of the counting problem, e.g., its relation with the number of real roots of the equations which describe the dark lines seen in bifurcation diagrams of the maps, or the number of saddle-nodes in forming the Smale horseshoes [12], or the number of independent solutions of the corresponding renormalization group equations associated with the period-n-tupling sequences (for details see [5]).

4. Number of Periods in Maps with Discontinuity

As there are infinitely many ways to introduce discontinuities into a one-dimensional map, it is difficult to obtain general counting results for such maps. However, we do have some closed results for two particular cases [11].

4.1. The Gap Map

The gap map is obtained by opening a gap at the top of a unimodal map, thus making it a two-parameter map. It can be shown that from each periodic orbit of the unimodal map, one gets four different periodic orbits as long as $n \geq 3$, i.e., its number of periods is given by $4N_2(n)$. For $n = 1$ and 2, there are 2 and 3 different periods as it can be seen by direct inspection.

4.2. The Lorenz-Type Map

By reversing the right branch of a unimodal map one gets the so-called Lorenz-type map which is instructive for the understanding of the chaotic dynamics in the Lorenz equations. Its number of periods is $2Z_2(n)$ for $n \geq 2$, where

$$Z_m(n) = \sum_{d|n} \varphi(d) m^{\frac{n}{d}}$$

is the number of periodic sequences which are invariant under the cyclic group of order n alone, as given in [2]. By inspection, one sees that there are two fixed points in this map.

5. Discussion

We have given complete solutions of how to count the number of periods in one-dimensional continuous maps of the interval and we have indicated partial results for a few maps with discontinuity. However, we must admit that the problem has been made much simpler by the way we pose it. In fact, we have been looking for periodic orbits in the entire parameter space. As an ordering may be introduced for symbolic sequences of any one-dimensional maps and the best way to parametrize a map with multiple critical points is to use its so-called *kneading sequences* [5] as parameters, one may put all possible parameter combinations into a one-dimensional sequence and then ask how many and what periodic orbits have appeared at a given parameter set. This turns out to be a much more difficult problem. So far only a limited result confined to the period 3 window of unimodal maps has been known [8].

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