On Statistical Field Theory

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Prologue: Dr. Ayse Erzan has been a long-time friend of Chinese statistical physics community. She paid a visit to the Institute of Theoretical Physics in Beijing almost 30 years ago, in the early days of the Institute when we worked in army barracks. She was the only participant from Turkey to the 19th IUPAP International Conference on Statistical Physics, held in Xiamen, China, in 1995. Both me and my colleague Dr. Weimou Zheng visited her in Istanbul. Taking the happy opportunity of celebrating Ayse's 60th birtrhday, I would like to make public a "private" letter I wrote to Ayse after my visit to the Gursey Institute in 1999. It was "private" in the sense that we continued our scientific discussion in a free and informal way without worry of committing possible mistakes. Therefore, I did not change a single word in what published below except for adding a title.

Happy Birthday to Ayse!

16 December 1999
Dr. Ayse Erzan
F. Gursey Enstitusu
Rasathane, Yolu
Gengelkoy, Istanbul Turkey

Dear Ayse,

This is a continuation of our discussion in November at the Gursey Institute. I have located the references and recollected something from the back corner of my memory. In fact, some description might be found in our Physics Reports (1985) review. However, I will do it in a very loose, hand-waving way, a way one cannot follow when writing an "official" review. Anyway, without thinking about a referee I will write quite freely and with possible mistakes.

Let us start from a Landau-Ginzburg (LG) type free energy F. The phenomenological theory of superconductivity, obtained from the equilibrium condition

$$\frac{\partial F}{\partial \Psi_i} = 0$$

was so good as to include the theory of the type II superconductors. Now we want to extend it to non-equilibrium situation, to a time-dependent LG (TDLG), by writing phenomenologically:

$$\frac{\partial \Psi_i}{\partial t} = -\sigma_{ij} \frac{\partial F}{\partial \Psi_j}.$$

If the coefficient matrix σ_{ij} is symmetric, it corresponds to irreversible relaxation process. After diagonalization, if Ψ_i are not conserved quantities, one may take σ_{ij} to be a constant and get a simple dissipative relaxation. If Ψ_i are conserved quantities, we may compare it with diffusion process and see that σ_{ij} may contain differential operators with respect to spatial coordinates, something like $-D_{ij}\nabla^2$, where $\mathrm{diag}(D_{ij})$ are diffusion coefficients. In this case dissipation is a high-order effect. If σ_{ij} is anti-symmetric, it may describe reversible canonical motion. In the simplest case σ_{ij} is a symplectic matrix and we have Hamiltonian canonical equations.

The dissipative nonlinear interactions among the Ψ_i modes may be taken into account by adding coupling terms in the LG free energy. It is these terms that make the kinetic coefficients vanishing at the critical point. However, not all interactions may be put into the free energy, e.g., the Landau-Lifshitz vector product in theory of magnetism. One of the important progresses on critical dynamics was the realization that some non-dissipative, reversible coupling may affect critical phenomena in an essential way, in particular, causing the kinetic coefficients to diverge at the critical point. In order to reflect this, one adds the Kawasaki mode-mode coupling terms:

$$V_i(\Psi) = \lambda \sum_j \left(\frac{\partial}{\Psi_j} A_{ij}(\Psi) - A_{ij}(\Psi) \frac{\partial F}{\partial \Psi_j} \right),$$

where an anti-symmetric tensor A_{ij} is composed of commutators or Poisson brackets made of Ψ_i. This term is also called a convective or a streaming term. The name 'streaming" comes from the fact that after transforming the TDGL into a generalized Langevin equation by adding random forces, $V_i(\Psi)$ becomes one of the flow terms in the corresponding Fokker-Plank equation. The very form of V_i guarantees that it satisfies the equation of conservation of probability:

$$\frac{\partial}{\partial \Psi_j}(V_j e^{-F}) = 0.$$

Now we have arrived at a generalized Langevin equation (GLE):

$$\frac{\partial \Psi_i}{\partial t} = K_i(\Psi) + \xi_i(t),$$

where the non-stochastic part is:

$$K_i(\Psi_i) = -\sigma_{ij}\frac{\partial F}{\partial \Psi_j} + V_i(\Psi).$$

The random force ξ_i takes into account the interactions among all degrees of freedom that were not included in Ψ_i. We know little about ξ except for assuming that they obey Gaussian distribution:

$$\begin{aligned}\langle \xi_i(t)\rangle &= 0,\\ \langle \xi_i(t)\xi_j(t')\rangle &= 2\sigma_{ij}\delta(t-t').\end{aligned}$$

If one keeps only the symmetric part of σ_{ij}, attributing the anti-symmetric part into A_{ij}, then the σ_{ij} that appears in the Gaussian correlation is the same. This is the requirement of the dissipation-fluctuation theorem.

Different choice of Ψ_i and their symmetry property as well as the concretization of F leads to different models of critical dynamics, enumerated in the Hohenberg-Halperin review by letters from A to J(?) (*Rev. Mod. Phys.* **49** (1977) 435 and our paper *Phys. Rev.* **B22** (1980) 3385. It was in this last reference the fact was pointed out that the mode-mode coupling is nothing but the Ward-Takahashi identity).

An usual way of dealing with the GLE is to take the nonlinear terms be small perturbations and solve the equations by iteration. This is done, e.g., in Chapter 14 of S. K. Ma's *Modern Theory of Critical Phenomena* (1976). In the expansion obtained in this way there are two kinds of constituent blocks — response functions and correlation functions. Therefore, their graphs differ from Feyman. If one tries to figure out what kind of higher terms should appear, one would see some series of graphs do not appear in Ma's expansion. This was understood later, when Georgio Parisi and Yong-shi Wu were working in Beijing on stochastic quantization of gauge theories. In a discussion, Georgio realized that it was caused by cancellation of terms coming from the Jacobian (see below). I don't think Georgio's remark has been put in record anywhere.

The situation reminds that in theory of turbulence. Krishnan developed a graphic expansion but could not get higher-order graphs correct. This was one of the motivation of the MSR field theory (*Phys. Rev.* **A8** (1973) 423). In fact, the main achievement of MSR was the correct enumeration of high-order graphs.

In both cases (GLE and MSR) one starts directly from the field equations and this has made it inconvenient to compare with quantum field theory. In a sense, these were Hamiltonian field theories and there was a need to construct a Lagrangian statistical field theory.

Now I come to the main point: Lagrangian statistical or stochastic field theory. The earliest work was the Onsager-Machlup probability density functional for linear Markovian processes (*Phys. Rev.* **91** (1953) 1505, 1512). The generalization to nonlinear probability density functional was obtained by R. Graham (*Springer Tracts in Modern Physics* **66** (1973) 1) and Kubo's group (R. Kubo, K. Matsuo, and K. Kitahara, *J. Stat. Phys.* **9** (1973) 51) almost at the same time, but Kubo ignored the Jacobian. Later on Janssen (*Z. Phys.* **B23** (1976) 377; B24 (1976) 113) and De Dominicis (*J. de Phys.* **37** (1976) Colloq. C-247; *Phys. Rev.* **B18** (1978) 353) applied it to critical dynamics.

Let me demonstrate all these development by "getting everything from nothing": the decomposition of the unit.

To guarantee the GLE we start from the normalization condition of a functional δ function:

$$\int [d\Psi]\delta\left(\frac{\partial \Psi}{\partial t} - K(\Psi) - \xi\right)\Delta(\Psi) = 1.$$

Since the argument of the δ-function is not Ψ but the whole expression $GLE = 0$, we must include a functional Jacobian $\Delta(\Psi)$. If one looks at the GLE as a transformation from a Gaussian stochastic processes $\{\xi_i\}$ to more complicated stochastic processes $\{\Psi_i\}$, Δ is the Jacobian of this transformation.

Graham first calculated this Jacobian. Up to some (infinite ?) constant it is

$$\Delta(\Psi) = exp\Bigg(-\frac{1}{2}\int \frac{\partial K(\Psi)}{\partial \Psi} dxdt\Bigg).$$

(We will neglect $dxdt$ in what follows.) Fortunately, this Jacobian has an exponential form, unlike that in gauge theory where one has to exponentiate it $exp^{\ln J}$, thus yielding a non-polynomial effective Lagrangian.

If one expresses the δ-function by a path integral over $\hat{\Psi}$, an auxiliary field, one gets

$$\int [d\Psi][\frac{\hat{\Psi}}{2\pi}]exp\left\{i\hat{\Psi}\left(\dot{\Psi} - K(\Psi) - \xi\right) - \frac{1}{2}\frac{\partial K}{\partial \Psi}\right\} = 1. \tag{1}$$

(This was the Fadeev-Popov trick in gauge theory.) Now insert a factor $exp(i\int(J\Psi + \hat{J}\hat{\Psi})dxdt)$ to transform the above normalization condition into a generating functional for all possible products of Ψ and $\hat{\Psi}$ (composite operators):

$$Z_\xi[J,\hat{J}] = \int [\frac{d\Psi\hat{\Psi}}{2\pi}]exp\left\{i\hat{\Psi}\left(\dot{\Psi} - K(\Psi) - \xi\right) - \frac{1}{2}\frac{\partial K}{\partial \Psi} + iJ\Psi + i\hat{J}\hat{\Psi}\right\}. \tag{2}$$

This is the characteristic or moment-generating functional in probability theory. It is also a functions of ξ. Obviously, $Z_\xi[0,0] = 1$. The ξ obeys a Gaussian distribution

$$W(\xi) \sim exp\left(-\xi\sigma_{ij}^{-1}\xi/2\right),$$

where σ^{-1} is the inverse of our familiar σ.

Performing the Gaussian integration over ξ, one gets:

$$Z[J,\hat{J}] = \int [\frac{d\Psi\hat{\Psi}}{2\pi}]exp\left\{-\frac{1}{2}\hat{\Psi}\sigma\hat{\Psi} + i\hat{\Psi}\left(\dot{\Psi} - K(\Psi))\right) - \frac{1}{2}\frac{\partial K}{\partial \Psi} + iJ\Psi + i\hat{J}\hat{\Psi}\right\}. \tag{3}$$

This is nothing but the Lagrangian generating functional for the MSR field theory with the "conjugate field" $\hat{\Psi}$ or response field. Like in quantum field theory these Ψ and $\hat{\Psi}$ under a path integral are classical commutative fields. The integration over $\hat{\Psi}$ is again a Gaussian. Taking the integral, we get

$$Z[J,\hat{J}] = \int [d\Psi] exp\left\{-\frac{1}{2}(\dot{\Psi}-K-\hat{J})\sigma^{-1}(\dot{\Psi}-K-\hat{J}) - \frac{1}{2}\frac{\partial K}{\partial \Psi} + iJ\Psi\right\}. \tag{4}$$

Eq. (4) was first obtained by Graham (with $\hat{J}=0$) as a probability density functional. It has two shortcomings:

1. The nonlinear term $K(\Psi)$ enters in a product, leading to higher nonlinearities and worse renormalizability (seen by power counting of the resulting field theory).
2. The inverse matrix σ^{-1} may lead to singularity in the case of conserved quantities, as after Fourier transform one gets $\sigma_i = D_i k^2$ from the Laplacian operator.

The starting point for the construction of a Lagrangian field theory is Eq. (3) where one deals with $K(\Psi)$ and σ themselves (Janssen, De Dominicis, *et al.*). The cost one pays is there is the conjugate field $\hat{\Psi}$ that doubles the degrees of freedom. In fact, it is a common feature of several formalisms on non-equilibrium statistical physics that one has somehow to deal with doubled degrees of freedom:

1. The superoperator formalism of Prigogine school and the like, see, e.g., M. Schmutz, *Z. Phys.* **B30** (1978) 97.
2. The closed-time path Greens functions (CTPGF) — by doubling the time (see the "Gang of Four", 1985).

Kubo and coworkers first got Eq. (3), but they lost the Jacobian, which is necessary to secure causality. With the Jacobian kept, a Feyman expansion would lead to the terms seen in S. K. Ma's work and graphs not seen in Ma's work was due to cancellation with what follows from the Jacobian.

In summary, Graham got the correct Jacobian but did the Gaussian integration too quickly; Kubo kept the Gaussian but lost the Jacobian.

Replacing $i\hat{\Psi}$ by $\hat{\Psi}$, one gets the Lagrangian:

$$L(\Psi,\hat{\Psi}) = \frac{1}{2}\hat{\Psi}\sigma\hat{\Psi} + \hat{\Psi}(\dot{\Psi}-K(\Psi)) - \frac{1}{2}\frac{\partial K}{\partial \Psi}. \tag{5}$$

Now one can carry over all the arsenal of quantum field theory to stochastic field theory – power counting and dimensional analysis, dimensional regularization, minimal renormalization, Callen-Symanzik equation, etc.

Nevertheless, all this remains a phenomenological approach, as it starts from the phenomenological GLE. CTPGF was an attempt to give it an *ab initio* flavor.

All I have described was an old, forgotten chapter of nonequilibrium statistical physics. I have not followed the new development since I turned to nonlinear dynamics. The only suggestions I would like to make were what I told you in Istanbul:

1. Deal with the Jacobian carefully. Your integration is over E, a scalar variable, but the field is a continuum. There should be a Jacobian at some stage. The cancellation, if any, may simplify your calculation.
2. Do not get ride of the Gaussian integration too early, otherwise it may complicate the subsequent calculation.

Please convey my best wishes for the New Millennium to all colleagues at the Gursey Institute whom we met at lunch time or seminars.

Sincerely yours,

Bailin Hao

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Note: Written in 1999, first published in *Ayse Erzan Olmak* $\cdots$, May 2009, Istanbul, pp. 17 – 21.