Review

International Journal of Modern Physics B
Vol. 28, No. 16 (2014) 1430008 (21 pages)

DOI: 10.1142/S0217979214300084

World Scientific
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Skeleton graph expansion of critical exponents in "cultural revolution" years

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Received 8 February 2014
Accepted 20 February 2014
Published 3 April 2014

Kenneth Wilson's Nobel Prize winning breakthrough in the renormalization group theory of phase transition and critical phenomena almost overlapped with the violent "cultural revolution" years (1966–1976) in China. An unexpected chance in 1972 brought the author of these lines close to the Wilson–Fisher ϵ-expansion of critical exponents and eventually led to a joint paper with Lu Yu published entirely in Chinese without any English title and abstract. Even the original acknowledgment was deleted because of mentioning foreign names like Kenneth Wilson and Kerson Huang. In this article I will tell the 40-year old story as a much belated tribute to Kenneth Wilson and to reproduce the essence of our work in English. At the end, I give an elementary derivation of the Callan–Symanzik equation without referring to field theory.

Keywords: Kenneth Wilson; renormalization group; ϵ-expansion of critical exponents; skeleton graph expansion; Callan–Symansik equation.

PACS numbers: 01.65.+g, 05.70.Fh, 11.10.Hi

1. Introduction

There has never been a tower of ivory for scientists working in a developing country. In his report to the South Commission[1] Abdus Salam reproduced a figure by Dadison Frame. The figure showed the annual publication of scientific and technological papers versus GNP for the year 1973. Most points fell around two straight lines, that for developed and developing countries, respectively. There was, however, a striking lonely outlier far below many least-developed countries. It represented China.

In the summer of 1972, an unthinkable opportunity threw four Chinese physicists into the Canadian Congress of Physicists held in Edmonton, Alberta. Michael Fisher talked about his joint work with Kenneth Wilson on ϵ-expansion of the critical exponents.[2] With my poor English at that time, I could only appreciate the importance of the renormalization group approach. Upon return to Beijing, I immediately read the two 1971 papers of Wilson.[3,4]

I must explain how could I get access to scientific literature in a time when all libraries were closed. At that time, the Institute of Physics was led by representatives of the People's Liberation Army (PLA). In order to prepare for "science reform" they appointed a group of scientists to do investigations on physics research in China and abroad. This allowed me to get into the closed libraries. Xerox machine was something unheard of. We had to make notes by hand.

The first two papers of Wilson were quite hard to grasp. Early 1973, Professor Kerson Huang paid a visit to Beijing. In a discussion, he mentioned that Kenneth Wilson had given a series of lectures in Princeton and promised to write to Kenneth. Soon, Wilson sent us a Cornell preprint[5] which later appeared in *Physics Reports*.[6] The "later" here meant at least half a year delay or more as the libraries had only surface mail subscriptions.

After digesting the available information, our goal was clear: calculate the ϵ expansion to high powers of ϵ in order to compare with experimental measurements and to check the scaling relations among critical exponents. An approach by comparing skeleton graph with scaling relation[7,8] came to my attention. Skeleton graphs were closer to my heart as more than 10 years ago in one of the Landau seminars in Moscow I listened to K. A. Ter-Martirosyan talking about applying skeleton graph expansion to meson scattering. The authors of Refs. 7, 8 considered the special case of $n = 2$ Bose systems and their result could not be extended to general n and higher-orders. Especially, the analysis at the critical point was lacking. I undertook to study the general n case.

There was a theoretical division in the Institute of Physics, of which I was the Deputy Head. The division was created in 1959 upon reflection of the "Great Leap Forward". In 1969, it was disbanded by the PLA officers as a "typical example of isolation of theory from practice". In 1973 I succeeded in restoring a small theoretical group in the laboratory of magnetism. Another important fact that allowed me to do some science consisted in that I was nailed to the bed by lumbar disc rupture. Our group member Lu Yu came and worked at my bedside and then reported to the group. Eventually, Yu was the only one who could catch up with the work.

The official journal of the Chinese Physical Society *Acta Physica Sinica* stopped publication for more than seven years, from the fall of 1966 to the end of 1973. It was decided to restore publication from January 1974. Our paper[9] arrived at the editorial office on 5 December 1973 and appeared in print only in May 1975. The paper did not have an English title and abstract. Originally, we thanked Kerson Huang and Kenneth Wilson for helping with the Princeton lecture notes. However, we were asked for the political attitude of these foreigners and we decided to delete the whole acknowledgment. In the spring of 1977, a group of solid state physicists from the Chinese Academy of Sciences visited France and Germany for the first time after many years of isolation from the outside world. After my talk at Orsay on closed-form approximation for the three-dimensional Ising model, I had a discussion with Eduard Brézin by pointing to formulas in our Chinese reprint. This was the start of our many-decade friendship with Eduard Brézin.

2. Critical Exponents and Scaling Relations

In continuous phase transitions, thermodynamic functions and their first derivatives are continuous, but high-order derivatives may be singular at the transition point T_c. The behavior of thermodynamic quantities near critical point is described by various critical exponents. For example, the singularity of specific heat near a critical point is described by the exponent α:

$$c_v \sim (T - T_c)^{-\alpha} \quad (T \geq T_c) \tag{1}$$

The behavior of spin or density correlation function near the critical point is better expressed via their Fourier transform as

$$G(P \to 0, T = T_c) \sim P^{-2+\eta}\,, \tag{2}$$

which involves another exponent η.

Another limit of the same correlation function is associated with the initial magnetic susceptibility (or isothermal compressibility) χ_γ:

$$\chi_\gamma \sim G(P = 0, T \to T_c + 0) \sim (T - T_c)^{-\gamma}\,, \tag{3}$$

where a third exponent γ is introduced.

Historically, various phase transition analyses were unified in the Landau mean field theory which yields the same exponents: $\alpha = 0$ (finite discontinuity), $\gamma = 1$, and $\eta = 0$.

The situation was more or less satisfactory until the mid 1960s when precise experimental measurements and exact statistical models all showed that there were definite deviations of critical exponent values from the mean field theory. Nevertheless, some relations between the critical exponents turned out to be holding. Leo Kadanoff[10] and Michael Fisher[11] called attention of the statistical physics community to this challenge. Unfortunately, their well-known reviews[10,11] were overlooked by Chinese physicists as the "cultural revolution" broke out on 1st June 1966.

3. Classical Field Theory Representation of Statistical Problem Near Critical Point

We adopt the model of Wilson.[4] Consider the interaction of classical spins with n components in a d-dimensional lattice. Here, "spin" is nothing but the order parameter in theory of continuous phase transition. In the calculation of statistical partition function, the summation goes over all states $\{S\}$ from nearest-neighbor lattice points

$$Z = \sum_{\{S\}} \exp\left(-\frac{E}{kT}\right) = \sum_{\{S\}} \exp\left(\frac{K}{2} \sum_{m,i} S_m^\alpha S_{m+i}^\alpha\right), \quad K = \frac{J}{kT}\,, \tag{4}$$

where J denotes the exchange integral, m represents the position vector of the cell, i represents the relative position vector of the nearest neighbors (in what follows we do not use boldface for vectors). The repeated spin superscript α performs summation from 1 to n. By introducing a convergent factor $-\frac{b}{2}S_m^\alpha S_m^\alpha$ and treating S_m^α as a variable taking continuous values, the summation in Eq. (4) may be replaced by integration. Furthermore, changing the lattice point function S_m to a function of continuous medium $S(x)$ and absorbing K by rescaling S, we get

$$Z = \left(\int \mathcal{D}S\right) \exp\left[-\frac{1}{2}\int_x ((\triangledown S)^2 + r_0 S^2)\right], \quad r_0 = \frac{b}{K} - 2d\,. \tag{5}$$

From now on, functional integration over infinite function system as well as ordinary d-dimensional integration will be represented by shorthand notations:

$$\left(\int \mathcal{D}S\right) \equiv \lim_{m\to\infty}\left(\prod_m \int_{-\infty}^{\infty} dS_m\right), \quad \int_x \equiv \int d^d x, \int_q \equiv \int \frac{d^d q}{(2\pi)^d}\,. \tag{6}$$

The gradient term $(\triangledown S)^2$ comes from nearest neighbor interaction. Performing Fourier transformation $S(x) = \int_q e^{iqx}\sigma_q$ and neglecting the common factor which appears during function substitution in the functional integral, we have

$$Z = \left(\int \mathcal{D}\sigma\right) \exp(H_0) \tag{7}$$

and

$$H_0 = -\frac{1}{2}\int_q (q^2 + r_0)\sigma_q^\alpha \sigma_{-q}^\alpha\,. \tag{8}$$

H_0 is the Hamilton function of a free classical field. It is the simple Gaussian model in statistics. All statistical averages are Gaussian averages. For example, spin correlation function is nothing but the propagator of the classical field:

$$\langle \sigma_1^\alpha \sigma_2^\beta \rangle = G_0(q_1, r_0)\delta(1+2)\delta_{\alpha\beta} = \frac{\delta(1+2)}{q_1^2 + r_0}\delta_{\alpha\beta}\,, \tag{9}$$

where $1, 2$ are shorthand for the momenta q_1, q_2. We know the temperature dependence near critical point from Eq. (3):

$$\chi_\gamma \sim G_0(0, r_0) = \frac{1}{r_0}, \quad r_0 \sim (T - T_c)^\gamma\,. \tag{10}$$

In the Gaussian model, all high even-order spin correlation functions decompose into combinations of second-order correlation functions, and odd-order correlation functions vanish. This decomposition corresponds to Wick theorem in field theory.

Wilson observed that critical exponents of Gaussian model coincide with that of mean-field theory. Spin distribution function in Gaussian model has a maximum at $S = 0$, far departing from general statistical models. However, if one inserts a fourth-order term into the convergence factor of Eq. (5), one may go beyond the mean field theory. Then, the Hamilton function reads:

$$H = H_0 + H_I = H_0 - \frac{u_0}{3}\Delta(\alpha\beta\gamma\delta)\int \sigma_1^\alpha \sigma_2^\beta \sigma_3^\gamma \sigma_4^\delta \delta(1+2+3+4)\,, \tag{11}$$

where u_0 corresponds to the coupling constant in field theory with ϕ^4 interaction, $\Delta(\alpha\beta\gamma\delta)$ is a fully symmetric unit tensor:

$$\Delta(\alpha\beta\gamma\delta) = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\,. \tag{12}$$

Various products and contractions of $\Delta\alpha\beta\gamma\delta$ (i.e., summations over repeated indices) frequently appear in the calculation of high-order terms. We postpone these monotonic yet useful technicalities to Sec, 6.2.

The average of any product A of field functions

$$\langle A\rangle = \frac{\langle A\exp(H_I)\rangle_0}{\langle \exp(H_I)\rangle_0}$$

may be decomposed and one can prove a connected-graph expansion theorem. Detailed enumeration of diagram coefficients and calculations of integrals will be given in Secs. 6 and 7.

In this model, propagators contain temperature, but bare interactions do not depend on temperature. High-order vertexes depend on temperature by way of propagators. Consequently, the momentum-independent part in self-energy diagram leads to a shift of critical point. In order to take into account this point, it is better to include this part of contribution from the self-energy diagram into r_0. This "mass renormalization" process may be realized by way of a cancelation term well-known in field theory, i.e., rewriting Eq. (11) to

$$H = -\frac{1}{2}\int (q^2+r)\sigma_1^\alpha\sigma_{-1}^\alpha - \frac{1}{2}\int (r_0-r)\sigma_1^\alpha\sigma_{-1}^\alpha - \frac{u_0}{3}\Delta(\alpha\beta\gamma\delta)\int \sigma_1^\alpha\sigma_2^\beta\sigma_3^\gamma\sigma_{-1-2-3}^\delta \tag{13}$$

and requiring that the exact propagator $G(q, r)$ satisfies a condition similar to Eq. (10) at $q \to 0$:

$$\chi_\gamma \sim G(0, r) = \frac{1}{r} \sim (T - T_c)^{-\gamma}\,. \tag{14}$$

Momemtun-independent part in self-energy diagram cancels out with the second term in Eq. (13), the cancelation equation defines r. When calculating a complex diagram containing self-energy part, one must introduce a subtraction term as explained in details in Eq. (54) in Sec. 6. In this paper, we use the same notation for the exact propagator and for the free propagator $G(q, r) = (q^2 + r)^{-1}$ after "mass renormalization", their difference is clear from the context. Skeleton graphs are those composed of exact propagators.

4. Skeleton Graph Analysis of Four-Point Vertex

The four-point vertex $\Gamma^{\alpha\beta\gamma\delta}(1234)$ discussed in this section is four-spin correlation function excluding disconnected diagrams and amputating single-particle external lines. Usually the r dependence is not written out explicitly. Since the σ's are commutative classical quantities, Γ is fully symmetric with respect to both spin

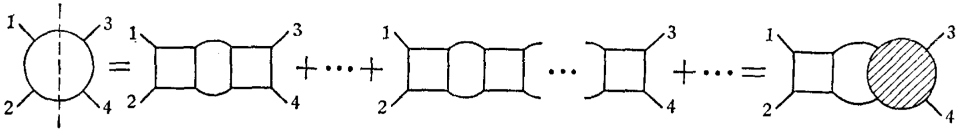

Fig. 1.

indices and momentum. We bind together the spin superscript and momentum, i.e., binding $1 \leftrightarrow \alpha$, $2 \leftrightarrow \beta$, etc. There are 4! permutations. In fact, they should be symmetrized separately, leading to $(4!)^2$ permutations. Later on, when discussing the forward scattering amplitude at $r = 0$, we have to symmetrize the spin superscripts separately.

The 4! diagrams obtained from a four-point vertex by permutating external lines are divided into three groups: $(12;34)$, $(13;24)$ and $(14;23)$, corresponding to the s, t and u channels in field theory. Some diagrams are reducible in one channel, i.e., becoming disconnected parts by cutting two internal lines; some are entirely irreducible. Diagrams reducible in one channel are irreducible in the other two channels. A complex diagram may be reducible in one channel, its subdiagrams may be reducible in other channels. If all subdiagrams are reducible in one or another channel, then the complex diagram is called a parquet diagram. The total of diagrams, reducible in one channel, is denoted as, e.g., $\Gamma^{\alpha\beta;\gamma\delta}(12;34)$. The total of diagrams, irreducible in one channel, is denoted as $I^{\alpha\beta;\gamma\delta}(12;34)$, represented by squares in Fig. 1. $\Gamma^{\alpha\beta;\gamma\delta}$ may be obtained from $I^{\alpha\beta;\gamma\delta}$ by iteration, as shown in Fig. 1.

If taking out the leftmost square and summing over the remaining I, we get a fully symmetrized total vertex $\Gamma^{\alpha\beta\gamma\delta}(-5-634)$:

$$\Gamma^{\alpha\beta;\gamma\delta}(12;34) = -36 \int I^{\alpha\beta;\mu\nu}(12;56)G(5)G(6)\Gamma^{\mu\nu\gamma\delta}(-5-634)\,. \tag{15}$$

δ-functions ensuring momentum conservation are not written out explicitly in the above formula. The calculation of the numerical coefficient is given in Sec. 6. The total vertex is expressed via the sum R of vertexes reducible in some channel and diagrams irreducible in all three channels, the lowest order diagram of the latter is the 4-point diagram in Fig. 3(h):

$$\begin{aligned}\Gamma^{\alpha\beta\gamma\delta}(1234) = {} & \frac{1}{3}(u_0 + R)\Delta(\alpha\beta\gamma\delta) + \frac{1}{3}\Gamma^{\alpha\beta;\gamma\delta}(12;34) \\ & + \frac{1}{3}\Gamma^{\alpha\gamma;\beta\delta}(13;24) + \frac{1}{3}\Gamma^{\alpha\delta;\beta\gamma}(14;23)\,. \end{aligned} \tag{16}$$

Inserting Eq. (15) into the above equation, we get Fig. 2.

$$\begin{aligned}\Gamma^{\alpha\beta\gamma\delta}(1234) = {} & \frac{1}{3}(u_0 + R)\Delta(\alpha\beta\gamma\delta) - 12 \int I^{\alpha\beta;\mu\nu}(12;56)G(5)G(6)\Gamma^{\mu\nu\gamma\delta}(-5-634) \\ & - 12 \int I^{\alpha\gamma;\mu\nu}(13;56)G(5)G(6)\Gamma^{\mu\nu\beta\delta}(-5-624)\end{aligned}$$

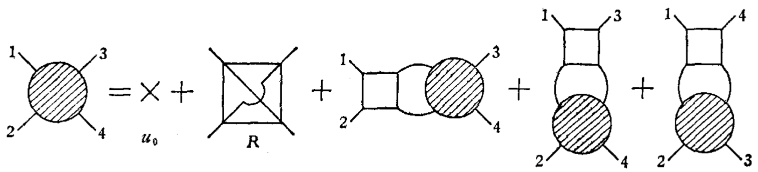

Fig. 2.

$$- 12 \int I^{\alpha\delta;\mu\nu}(14; 56)G(5)G(6)\Gamma^{\mu\nu\beta\gamma}(-5 - 623) . \tag{17}$$

If separating out a part, irreducible in one channel:

$$I^{\alpha\beta;\gamma\delta}(12; 34) = \frac{1}{3}(u_0 + R)\Delta(\alpha\beta\gamma\delta) - 12 \int I^{\alpha\gamma;\mu\nu}(13; 56)G(5)G(6)\Gamma^{\mu\nu\beta\delta}(-5-624)$$
$$- 12 \int I^{\alpha\beta;\mu\nu}(14; 56)G(5)G(6)\Gamma^{\mu\nu\beta\delta}(-5-623) , \tag{18}$$

then Eq. (17) may be written as an ordinary Bethe–Salpeter equation:

$$\Gamma^{\alpha\beta\gamma\delta}(1234) = I^{\alpha\beta;\gamma\delta}(12; 34) - 12 \int I^{\alpha\beta;\mu\nu}(12; 56)G(5)G(6)\Gamma^{\mu\nu\gamma\delta}(-5 - 634) . \tag{19}$$

One can write down similar equations using I irreducible in the other two channels.

In the theory of critical phenomena, we have to discuss two limits of the total four-point vertex:

(1) The long wave length limit $P = 0$ which becomes important due to divergence of the correlation length. We need the vertex U_R near the critical point $r \to 0$:

$$\Gamma^{\alpha\beta\gamma\delta}(0000; r) \equiv \frac{1}{3}U_R\Delta(\alpha\beta\gamma\delta) . \tag{20}$$

(2) The forward scattering amplitude $\Gamma(P)$ at the critical point $r = 0$ in the $P \to 0$ limit:

$$\Gamma^{\alpha\beta\gamma\delta}\left(\frac{P}{2}\frac{P}{2}; -\frac{P}{2} - \frac{P}{2}\right) = \frac{1}{3}\Gamma(P)\Delta(\alpha\beta\gamma\delta) \tag{21}$$

By a skeleton graph analysis we can obtain $\frac{\partial U_R}{\partial r}$ and $\frac{\partial \Gamma(P)}{\partial P}$ from the above definitions. Further comparison with the scaling relation[6]

$$\frac{\partial U_R}{\partial r}/U_R = \frac{4 - d - 2\eta}{2 - \eta}\frac{1}{r} \tag{22}$$

and

$$\frac{\partial \Gamma(P)}{\partial P}/\Gamma(P) = (4 - d - 2\eta)/P , \tag{23}$$

one determines entirely U_R and $\Gamma(P)$ contained in the expression. A pivotal point here consists in that u_0 is not required to be a small quantity beforehand, but U_R

and $\Gamma(P)$ are indeed small when the physical system approaches four dimension (small ϵ) or the internal degree of freedom n is great.

We first consider the skeleton graph expansion of U_R.

For the sake of clarity, we first keep terms up to U_R^3. Putting external momentum to zero and performing "graphical differentiation": i.e., first take derivative of propagators in between the irreducible parts, due to the arbitrariness of their positions the infinite series on both sides again sum up to Γ; then take derivatives of I at the two ends and in the middle. In this way, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial r}\Gamma^{\alpha\beta\gamma\delta}(0)\\
&\quad= -36\int \Gamma^{\alpha\beta\mu\nu}(00q-q)\frac{\partial}{\partial r}\left(G^2(q)\right)\Gamma^{\mu\nu\gamma\delta}(-qq00)\\
&\qquad -36\int \Gamma^{\alpha\beta\mu\nu}(00q-q)G^2(q)\frac{\partial}{\partial r}I^{\mu\nu\gamma\delta}(-qq;00)\\
&\qquad -36\int \frac{\partial}{\partial r}\left(I^{\alpha\beta;\mu\nu}(00;q-q)\right)G^2(q)\Gamma^{\mu\nu\gamma\delta}(-qq00)\\
&\qquad +432\int \Gamma^{\alpha\beta\gamma\delta}(00q-q)G^2(q)\\
&\qquad \times\frac{\partial}{\partial r}\left(I^{\mu\nu;\rho\tau}(-qq;k-k)\right)G^2(k)\Gamma^{\rho\tau\gamma\delta}(-kk00)\,. \qquad (24)
\end{aligned}$$

It is clear from Eq. (17) that, up to terms of order U_R^2, the difference between $\Gamma(00q-q)$ and $\Gamma(0)$ consists only in the momentum transfer between vertexes. Therefore, we have

$$\begin{aligned}
\Gamma^{\alpha\beta\mu\nu}(00q-q) &\approx \frac{1}{3}U_R\Delta(\alpha\beta\mu\nu)\\
&\quad -\frac{4}{3}U_R^2(I(q,r)-I)\left(\Delta_2(\alpha\mu;\beta\nu)+\Delta_2(\alpha\nu;\beta\mu)\right)\,, \qquad (25)
\end{aligned}$$

where we have made use of the tensor contraction formulas and integration symbol given in Secs. 6 and 7. As long as the required order is reached, one may replace a vertex by U_R and take it off the integral; the contraction of spin indices in the vertexes leads to the corresponding symmetric tensors. In order to calculate higher order expansions, one iterates until the corresponding skeleton graph appears and then replaces it by U_R and then contract the spin indices. All the calculations below are done in this way. According to Eq. (18), the derivative of I is

$$\frac{\partial}{\partial r}I^{\alpha\beta;\mu\nu}(00;q-q) = -\frac{4}{3}U_R^2\frac{\partial}{\partial r}(I(q,r))\left(\Delta_2(\alpha\mu;\beta\nu)+\Delta_2(\alpha\nu;\beta\mu)\right)\,. \qquad (26)$$

Inserting the above two formulas into Eq. (24) and neglecting the last term of order U_R^4 in Eq. (24), we obtain

$$\frac{\partial}{\partial r}\Gamma^{\alpha\beta;\gamma\delta}(0) = -4\Delta_2(\alpha\beta;\gamma\delta)U_R^2 I' + 64\tau_3(\alpha\beta;\gamma\delta)U_R^3(I_b' - II').$$

Combining similar formulas in three channels is equivalent to carrying out symmetrization. The result reads

$$\frac{U_R'}{U_R} = -4(n+8)U_R I' + 64(5n+12)U_R^2(I_b' - II')\,. \tag{27}$$

Calculating the integrals at dimension $d = 4 - \epsilon$ and comparing with the scaling relation Eq. (22), we obtain the renormalized vertex:

$$U_R = \frac{2\pi^2}{n+8}\epsilon\left[1 + \frac{\epsilon}{2}\left(\frac{6(3n+14)}{(n+8)^2} + C - \ln\frac{4\pi}{r}\right)\right]. \tag{28}$$

In calculating the above formula we have used the critical exponent η up to the order ϵ^2, easily obtainable from $\Gamma(P)$ to the order ϵ. C in Eq. (28) is the Euler constant. If we do not assume the smallness of ϵ, U_R is still inversely proportional to $(n+8)$; therefore, U_R remains a first-order small quantity as long as the internal freedom n is great.

In order to get the next order terms, we have to keep U_R^4 terms and take into account the contribution of irreducible skeleton graph in Fig. 3(h). The result reads

$$\begin{aligned}
U_R' = {} & -4(n+8)U_R^2 I' + 64(5n+22)U_R^3\int\left[\frac{\partial}{\partial r}G^2(q)(I(q,r)-I) + G_2(q)\frac{\partial}{\partial r}I(q,r)\right] \\
& - 256(n^2+20n+60)U_R^4\int\left[\frac{\partial}{\partial r}G_2(q)(I(q,r)-I)^2 + 2G(q)^2(I(q,r)-I)\frac{\partial}{\partial r}I(q,r)\right] \\
& - 256(3n^2+22n+56)U_R^4\int\left[\frac{\partial}{\partial r}G^2(q)(I(q,r)-I)^2 + 2G^2(q)(I(q,r)-I)\frac{\partial}{\partial r}I(q,r)\right] \\
& - 1024(n^2+20n+60)U_R^4\int\left\{\frac{\partial}{\partial r}G^2(q)(I(q,r)-I)G(k)(G(k+q)-G(k))\right. \\
& \left. + \; G^2(q)\left[\frac{\partial}{\partial r}(G(k)G(k+q))(I(q,r)-I) + G(k)G(k+q)\frac{\partial}{\partial r}I(k,r)\right]\right\} \\
& - 128(3n^2+22n+56)U_R^4\int\left[2\frac{\partial}{\partial r}G^2(q)(I(k+g,r) - I(k,r))G^2(k)\right. \\
& \left. + \; G^2(q)G^2(k)\frac{\partial}{\partial r}I(k+q,r)\right] - 1536(5n+22)U_R^4 I_h'\,.
\end{aligned} \tag{29}$$

The terms in the above formula correspond to subgraphs (a), (b), (c_1), (c_2), (f), (g) and (h) in Fig. 3. Comparing the graph and integral term by term, we can summarize the rule of differentiation of skeleton graphs as follows: take derivatives of the double propagators one by one and from the propagators that are not being differentiated subtract a term which equals the zero-momentum term of that being

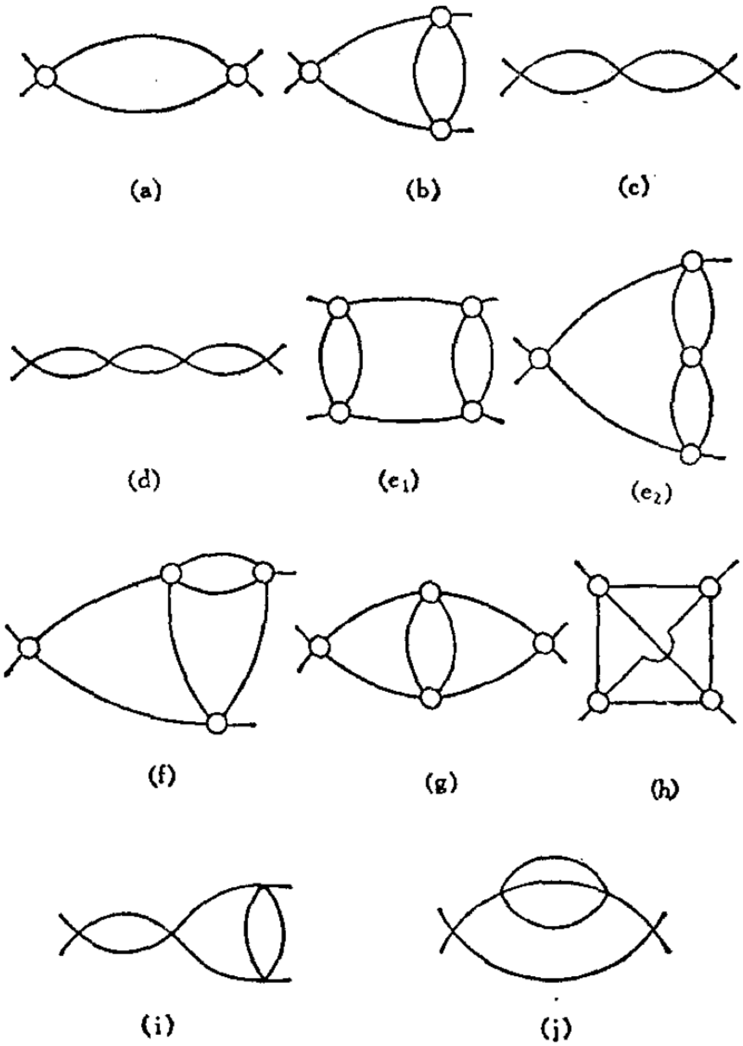

Fig. 3.

differentiated. For example, the second term corresponds to subgraph Fig. 3(b), when differentiating the first pair of reducible lines replace the right-side ring by $I(q,r) - I$.

Merging and putting in order the integrals in Eq. (29), we introduce the following symbols: $A = rI'$, $B = r(I'_b - II')$, $E = r(I'_e - 2II'_b + I^2 I')$, $F = r(I'_f - (II_b)' + I^2 I')$. $G = r(I'_g - 2I'I_b)$ and $H = rI'_h$. Integrals in the parentheses here may be divergent logarithmically. However, each symbol is a combination independent of the momentum cut-off. Therefore, U_R and all quantities derived from it are explicitly independent of the cut-off. Using these symbols, we may write down the next order expansion that includes Eq. (27) as:

$$\begin{aligned}\frac{rU_R'}{U_r} &= \frac{(4-d-2\eta)}{2-\eta} \\ &= -4(n+8)U_R A + 64(5n+22)U_R^2 B \\ &\quad - 512(2n^2+21n+58)U_R^3 E - 1024(n^2+20n+60)U_R^3 E \\ &\quad - 128(3n^2+22n+56)U_R^3 G - 1536(5n+22)U_R^3 H\,.\end{aligned} \tag{30}$$

Now we consider the skeleton graph expansion of $\Gamma(P)$.

The analysis of $\Gamma(P)$ is similar to that for U_R. However, one must keep in mind that in terms of momentum only one of the three channels describes forward scattering and symmetrization must be carried out with respect to spin indices. Therefore, in the definition Eq. (21) momentum and spin are not permutated together: there is only one channel for momentum, but for spin indices there are three combinations, i.e.,

$$\begin{aligned}\Gamma^{\alpha\beta\gamma\delta}\left(\frac{P}{2}\frac{P}{2};-\frac{P}{2}-\frac{P}{2}\right) &= \frac{1}{3}(u_0+R)\Delta(\alpha\beta\gamma\delta) + \frac{1}{3}\left[\Gamma^{\alpha\beta;\gamma\delta}\left(\frac{P}{2}\frac{P}{2};-\frac{P}{2}-\frac{P}{2}\right)\right. \\ &\quad \left. + \ \Gamma^{\alpha\gamma;\beta\delta}\left(\frac{P}{2}\frac{P}{2};-\frac{P}{2}-\frac{P}{2}\right) + \Gamma^{\alpha\delta;\beta\gamma}\left(\frac{P}{2}\frac{P}{2};-\frac{P}{2}-\frac{P}{2}\right)\right]\,.\end{aligned} \tag{31}$$

The rule of graphical differentiation of P is similar to that for r. Then, using expansions similar to Eqs. (25) and (26) and comparing with the scaling relation Eq. (23), we get

$$\frac{P\Gamma'(P)}{\Gamma(P)} = 4-d-2\eta = -4(n+8)\Gamma(P)A_P + 64(5n+22)\Gamma^2(P)B_P\,, \tag{32}$$

where $A_P = PI'(P)$, $B_P = P\left(I_b' - I(P)I'(P)\right)$. Carrying out ϵ expansion to ϵ^2, we have

$$\Gamma(P) = \frac{2\pi^2\epsilon}{n+8}\left[1+\frac{\epsilon}{2}\left(\frac{6(3n+14)}{(n+8)^2} + C - 2 - \ln\frac{4\pi}{P^2}\right)\right]\,. \tag{33}$$

The last formula should be compared with Eq. (28).

So far in the derivation of Eqs. (30) and (32) no assumption has been made concerning the spatial dimension d and the physical nature of smallness of U_R and $\Gamma(P)$, and no property of the bare coupling constant u_0 has been used. If taking u_0 as a small quantity we can directly write down perturbation expansions for U_R and $\Gamma(P)$, then insert u_0 back as expansion of U_R (or $\Gamma(P)$), the results would be identical to Eqs. (30) and (32). Due to the logical weakness of the necessity to require the smallness of u_0 and the lack of a natural way to get integral combinations that are independent of momentum cut-off, we prefer skeleton graph expansions to the "perturbation expansions" mentioned here.

5. Calculation of Critical Exponents

Using the renormalized vertex U_R at $p = 0$, one may calculate the critical exponents γ and α, from the forward scattering amplitude $\Gamma(P)$ at $r = 0$ one may get the exponent η.

First, consider the calculation of γ from the definition Eq. (14) of r after the "mass renormalization"

$$r = G^{-1}(0, r) = r_0 - \Sigma(0, r) \sim (r_0 - r_{0c})^\gamma \sim (T - T_c)^\gamma \,,$$

it follows that if we introduce a "three-point vertex":

$$\Lambda_0 \equiv \Lambda_0(0, r) \equiv \frac{dr}{dr_0} \sim \gamma(r_0 - r_{0c})^{\gamma-1} = \gamma r^{\frac{\gamma-1}{\gamma}} \,,$$

then on the one hand we have

$$\frac{\Lambda_0'}{\Lambda_0} = \frac{1}{r}\left(1 - \frac{1}{\gamma}\right) \,,$$

on the other hand we get

$$\Lambda_g = 1 - \Lambda_0 \frac{\partial}{\partial r}\Sigma(0, r) \,,$$

where $\Sigma(0, r)$ is the momentum-independent contribution from the self-energy part made of exact propagators. Differentiating the self-energy part, or equivalently, replacing every propagator G by $-G^2$ one after another, i.e., attach to each line a "photon" line with zero-momentum exchange. The three-point vertex comprises of the sum of all these diagrams and satisfies the Bethe–Salpeter equation (see Fig. 4):

$$\Lambda_0^{\alpha\beta}(p, r) = \delta_{\alpha\beta} - 12 \int I^{\alpha\beta\mu\nu}(p - p; k - k) G^2(k) \Lambda_0^{\mu\nu}(k, r), \tag{34}$$

where $\Lambda_0^{\alpha\beta} = \Lambda_0(p, r)\delta_{\alpha\beta}$. Performing graphical differentiation in the same way as we did in the previous section:

$$\begin{aligned} \frac{\partial}{\partial r}\Lambda_0^{\alpha\beta}(0, r) = &-12 \int \frac{\partial}{\partial r}(I^{\alpha\beta;\mu\nu}(00; k - k)) G^2(k) \Lambda_0^{\mu\nu}(k, r) \\ &-12 \int \Gamma^{\alpha\beta\mu\nu}(00; k - k) \frac{\partial}{\partial r} G^2(k) \Lambda_0^{\mu\nu}(k, r) \\ &+144 \int \Gamma^{\alpha\beta\mu\nu}(00; k-k) G^2(k) \frac{\partial}{\partial r}(I^{\mu\nu;\gamma\delta}(k-k; q-q) G^2(q) \Lambda_0^{\gamma\delta}(q, r)) \,. \end{aligned}$$

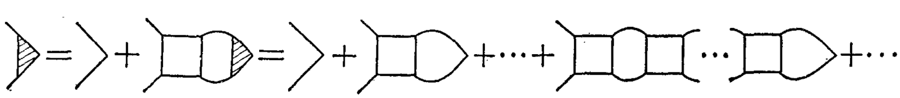

Fig. 4.

Inserting Eqs. (25), (26), etc, into the above formula, iterating $\Lambda_0(k,r)$ once in Eq. (34), and neglecting Λ_0 and k dependence in higher-order terms, we combine integrals and contract indices to get

$$\frac{r\Lambda_0'}{\Lambda_0} = 1 - \frac{1}{\gamma} = -4(n+2)U_R A + 96(n+2)U_R^2 B - 128(n+2)(n+8)U_R^3 E \\ - 512(n+2)(n+8)U_R^3 F - 384(n+2)^2 U_R^3 G \,. \qquad (35)$$

This formula maybe obtained by using the purely "perturbation theory" method mentioned at the end of the previous section. Multiplying Eq. (30) by $(n+2)/(n+8)$ and subtracting Eq. (35), and inserting the U_R obtained by solving Eq. (30), we get

$$1 - \frac{1}{\gamma} = \frac{n+2}{n+8}b + \frac{2(n+2)(7n+20)}{(n+8)^3}\frac{Bb^2}{A^2} - \frac{2(n+2)b^3}{(n+8)^4 A^3}\Big[(7n^2+68n+168)E \\ + 4(n^2+24n+56)F - 8(n-1)G + 12(5n+22)H \\ - \frac{8(5n+22)(7n+20)}{n+8}\frac{B^2}{A}\Big] \,, \qquad (36)$$

where $b = (4-d-2\eta)/(2-\eta)$. So far we have only used the fact that U_R is a small quantity without digging into its origin. Therefore, Eq. (36) has a wider application than the ϵ expansion. For the ϵ expansion, just insert the integrals and using the expression (43) for η, we get

$$1 - \frac{1}{\gamma} = \frac{n+2}{2(n+8)}\epsilon + \frac{3(n+2)(n+3)}{(n+8)^3}\epsilon^2 \\ + (n+2)\left[\frac{55n^2+268n+424}{2(n+8)^5} - \frac{18\zeta(3)(5n+22)}{(n+8)^4}\right]\epsilon^3 \,. \qquad (37)$$

$\zeta(3)$ in the above formula is the Riemann ζ function. It is worth mentioning that Eq. (37) is an expansion both in ϵ and in $\frac{1}{n}$. In Eq. (36) the coefficients of b^2 and b^3 terms do not contain zeroth order of n. After calculating the integrals explicitly, the $\frac{\epsilon^3}{n}$ terms in Eq. (37) cancel out. Whether the cancelation of the corresponding powers of $\frac{1}{n}$ continues in high orders requires further study. When the above results were obtained in 1973 there appeared ϵ^3 terms of γ calculated by other methods[13] without revealing the details.

Using the three-point vertex, one can define the polarized ring which is proportional to specific heat[14]:

$$c_v \sim (r_0 - r_{0c})^{-\alpha} \sim r^{-\frac{\alpha}{\gamma}} \sim \Pi(r) = n \int G^2(p)\Lambda_0(p,r) \,. \qquad (38)$$

The right-hand side of the above formula may be calculated from skeleton graph expansion

$$\Lambda_0(p,r) = \Lambda_0(0,r) + 96(n+2)U_R^2 \int G^2(q)(I(p+q,r) - I(q,r))\Lambda_0(q,r) \,.$$

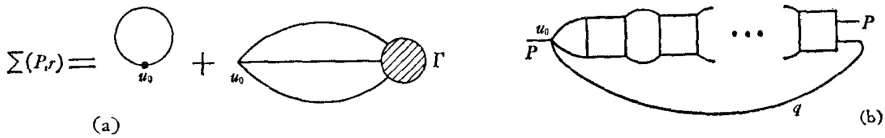

Fig. 5.

In fact, this formula can easily be derived from Eq. (34). Then compare the result with the consequence of the left-hand side

$$\frac{\Pi''}{\Pi'} = -\frac{1}{r}\left(1 + \frac{\alpha}{\gamma}\right), \tag{39}$$

we obtain

$$1 + \frac{\alpha}{\gamma} = -\frac{rI''}{I'} + 8(n+2)U_R A - 96(n+2)U_R^2 r\left[\frac{I_g''}{I'} - \frac{I_g' I''}{(I')^2} - 2II'\right]. \tag{40}$$

When calculating the first term one should pay attention to the contribution of Fig. 3(a) and (j). In the ϵ expansion there appears two momentum-independent combination of integrals. The final result reads

$$\frac{\alpha}{\gamma} = \frac{4-n}{2(n+8)}\epsilon + \frac{n+2}{(n+8)^2}\left[\frac{5}{2} - \frac{3(3n+14)}{n+8}\right]\epsilon^2. \tag{41}$$

Similar to Eq. (37), this formula is an expansion both in ϵ and $\frac{1}{n}$.

In order to calculate η, let $r = 0$ in the Dyson equation $G^{-1} = p^2 + r - \Sigma(p, r)$ and compare with Eq. (2), we get $p^{2-\eta} = p^2 - \Sigma(p, 0)$. For four-point interaction the irreducible self-energy Σ may be expressed via the four-point vertex,[12] see Fig. 5(a). Writing down the spin indices explicitly and ignoring terms independent of p, the Dyson equation may be rewritten as

$$p^{2-\eta}\delta_{\alpha\beta} = p^2\delta_{\alpha\beta} - 32u_0\Delta(\alpha\gamma\delta\rho)\int G(p-k)G(k-q)G(q)\Gamma^{\beta\gamma\delta\rho}(p-k, k-q, q, -p).$$

Now we perform graphical differentiation in the same way as in the previous section. Since the rightmost term is not I but the bare vertex u_0, see Fig. 5(b), there is no term corresponding to the third term in Eq. (24). Furthermore, the difference between I and u_0 should be deducted from the first term. From Eq. (18) we have

$$I^{\alpha\gamma;\delta\rho}(p, -q; k-p, q-k) = \frac{u_0}{3}\Delta(\alpha\gamma\delta\rho) - \frac{4}{3}\Gamma^2(p-q)[I(k)\Delta_2(\alpha\delta;\gamma\rho) + I(p+q-k)\Delta_2(\alpha\rho;\gamma\delta)].$$

Unlike the $r \neq 0$ and $p = 0$ limit, $\Gamma(p-q)$ must be kept within the integral. Putting in order and performing the contraction, we obtain

$$(2-\eta)p^{1-\eta} = 2p - 32(n+2)\int G(q)\frac{\partial}{\partial p}I(p-q)\Gamma^2(p-q)$$

$$+\frac{128(n+2)(n+8)}{3}\int G(q)\Gamma^3(p-q)\frac{\partial}{\partial p}\,[G(p-q)G(k-q)$$
$$\times\,(I(k)+I(p+q-k))-2I^2(p-q)]\,. \tag{42}$$

If the first term in the expansion of $\Gamma(p)$ does not depend on p (this is the case in the ϵ expansion, see Eq. (33)), the differentiation with respect to p in the last term may be taken out of the integral. Performing variable substitution, the two terms in the square brackets cancel out and Eq. (42) looks formally as an expansion to ϵ^2. Inserting the expansion Eq. (33) and performing the integration, then compare with

$$\frac{[G^{-1}(p)]''}{[G^{-1}(p)]'}=\frac{1-\eta}{p}\,,$$

which follows from Eq. (2), we get

$$\eta=\frac{n+2}{2(n+8)^2}\epsilon^2+\frac{n+2}{2(n+8)^2}\left[\frac{6(3n+14)}{(n+8)^2}-\frac{1}{4}\right]\epsilon^3\,. \tag{43}$$

In the process of calculating η, we did not define a "vector three-point vertex" $\Lambda_i=\frac{\partial}{\partial p}G^{-1}$ analogous to Λ_0, because it does not satisfy a Bethe–Salpeter equation similar to Eq. (34).

6. The Coefficients of Diagrams

The calculation of diagram coefficients includes $n=1$ and $n>1$ cases, the latter may be obtained from the former. For the parquet diagrams which are the basis of skeleton graphs we develop a method to calculate the coefficients of complex graphs.

6.1. *Coefficients of $n=1$ diagrams*

The coefficient $C(1)$ of a kth-order diagram at $n=1$ is the total number of diagrams with k vertexes and the corresponding internal and external lines. Its value is

$$C(1)=\frac{N_1N_2}{k!}\,, \tag{44}$$

where combinatorial factor N_1 is the number of diagrams at a fixed labeling of vertexes, topological factor N_2 is the number of nonequivalent diagrams obtained by permutation of the labels. In order to calculate N_1 we introduce a matrix representation of diagrams, suggested by our colleague Dr. Fuque Pu. Each kth-order diagram corresponds to a symmetric nonnegative real matrix, whose element a_{ij} denotes the number of internal lines between vertexes i and j, the diagonal element a_{ii} is twice the number of loops at the vertex i. A connected diagram corresponds to an irreducible matrix. Suppose vertex i comes from a W_i-point vertex, then the number of external lines is given by

$$f_i=W_i-\sum_{j=1}^{k}a_{ij}\geq 0\,,$$

Consequently,

$$N_1 = \frac{1}{a_{11}!a_{12}!\cdots a_{1k}!a_{22}!\cdots a_{2k}!\cdots a_{kk}!}\prod_{j=1}^{k}\frac{W_i!(a_{ii}-1)!!}{f_i!}\,. \tag{45}$$

The topological factor is obtained from a case by case analysis and no general algorithm has been formulated. For example, $N_1 = 72$, $N_2 = 1$, $C_a(1) = 36$ for Fig. 3(a); $N_1 = 3456$, $N_2 = 3$, $C_b(1) = 1728$ for Fig. 3(b); $N_1 = (4!)^4$, $N_2 = 3$, $C_h(1) = 41472$ for the nonparquet diagram Fig. 3(h).

6.2. *Products and contractions of fully and partially symmetric unit tensors*

In Eq. (11), we have used the fully symmetric unit tensor $\Delta^{\alpha\beta\gamma\delta}$ defined in Eq. (12). The products and contractions (i.e., summation over repeated indices) of $\Delta(\alpha\beta\gamma\delta)$ appear frequently in the $n > 1$ high-order diagrams. The symmetry property of any four-point diagram with fixed spin indices on external lines may be expressed as $A\delta_{\alpha\beta}\delta_{\gamma\delta} + B\delta_{\alpha\gamma}\delta_{\beta\delta} + C\delta_{\alpha\delta}\delta_{\beta\gamma}$, corresponding to the three channels mentioned before. There are in total three possibilities of indices permutation in this relation: $A = B = C$, fully symmetric, denoted as $(\alpha\beta\gamma\delta)$; $A \neq B = C$ (or similar cases), invariant with respect to indices permutation within each group or to exchange of two groups, denoted as $(\alpha\beta;\gamma\delta)$; $A \neq B \neq C \neq A$, denoted as $(\alpha,\beta;\gamma,\delta)$, invariant with respect to simultaneous permutation of indices within each group and ordered permutation of the two groups. All vertexes in the previous sections as well as symbols used below are written by using these notations. All products and contractions may easily be calculated. For example,

$$\begin{aligned}
\Delta_k(\alpha\beta;\gamma\delta) &= \Delta(\alpha\beta;\mu\nu)\Delta_{k-1}(\mu\nu;\gamma\delta) = \Delta_1(\alpha\beta;\mu\nu)\Delta_{k-1}(\mu\nu;\gamma\delta)\\
&= A_k\delta_{\alpha\beta}\delta_{\gamma\delta} + 2^{k-1}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,,\\
\Delta_k(\alpha\gamma;\beta\gamma) &= \left[A_k + 2^{k-1}(n+1)\right]\delta_{\alpha\beta}\,,
\end{aligned}$$

where $A_k = [(n+2)^k - 2^k]/n$, which may be derived by using induction. Skipping various general expressions, we list those used in previous sections in Table 1.

$P(n)$ in the last column of Table 1 comes from symmetrization. Summing over all possible permutations of indices and dividing by the number of permutations, the result is proportional to $\Delta(\alpha\beta\gamma\delta)/3$ with coefficient $P(n)$ being a polynomial of n. It is easy to see that when $n > 1$ $P(n)$ enters into the coefficients of the corresponding diagrams. For example, Fig. 3(a) is obtained from the contraction of two bare vertexes

$$\Delta(\alpha\beta\mu\nu)\Delta(\mu\nu\gamma\delta) = \Delta_2(\alpha\beta;\gamma\delta)\,,$$

and it corresponds to one channel, another channel obtained by permutation appears in Fig. 3(b):

$$\Delta(\alpha\beta\mu\nu)\Delta_2(\mu\gamma;\nu\beta) = \tau_3(\alpha\beta;\gamma\delta)\,.$$

Table 1. Symmetric tensors and their contractions.

Tensor	In Fig. 3	A	B	C	$P(n)$
$\Delta \equiv \Delta_1(\alpha\beta\gamma\delta)$	(h)	1	1	1	3
$\tau_2 = \Delta_2(\alpha\beta;\gamma\delta)$	(d), (j)	$n+4$	2	2	$n+8$
$\Delta_3(\alpha\beta;\gamma\delta)$	(c)	$n^2+6n+12$	4	4	$n^2+6n+20$
$\Delta_4(\alpha\beta;\gamma\delta)$	(d)	$n^3+8n^2+24n+32$	8	8	$n^3+8n^2+24n+48$
$\tau_3(\alpha\beta;\gamma\delta)$	(b)	$3n+10$	$n+6$	$n+6$	$5n+22$
$\tau_4(\alpha\beta;\gamma\delta)$	(e_2)	$n^2+10n+24$	$n^2+6n+16$	$n^2+6n+16$	$3n^2+22n+56$
$\rho_4(\alpha,\beta;\gamma,\delta)$	(e_1)	$8n+24$	$n^2+8n+20$	$4n+16$	$n^2+20n+60$
$\omega_4(\alpha\beta;\gamma\delta)$	(g), (i)	$3n^2+18n+32$	$2n+12$	$2n+12$	$3n^2+22n+56$
$\chi_4(\alpha\beta;\gamma\delta)$	(f)	$n^2+12n+28$	$4n+16$	$4n+16$	$n^2+20n+60$

In order to obtain the total coefficient of all three channels one carries out symmetrization and produces the corresponding $P(n)$. For a kth-order diagram $P(1) = 3^k$, because each bare vertex at $n > 1$ has three ways of spin propagation, corresponding to the three channels in Eq. (12). Every kth order diagram becomes 3^k diagrams and each spin loop brings about $\delta_{\alpha\alpha} = n$. Therefore, the coefficient of an $n > 1$ diagram is

$$C(n) = C(1)P(n)/3^k\,. \tag{46}$$

Examples: $C_a(n) = 4(n + 8)$; $C_b(n) = 64(5n + 22)$. The coefficient of nonparquet diagram Fig. 3(h) must be calculated directly and it is

$$\Delta(\alpha\mu\nu\rho)\Delta(\beta\rho\omega\sigma)\Delta(\gamma\mu\tau\omega)\Delta(\delta\tau\nu\sigma) = 3(5n + 22)\Delta(\alpha\beta\gamma\delta)\,.$$

Therefore, $C_h(n) = 1536(5n + 22)$.

6.3. *Parquetry rules*

We first consider the case $n = 1$. Suppose that a complex diagram decomposes into two subdiagrams (x) and (y) according to the reducible channels, as given in Fig. 6 with symmetry property of the diagram shown symbolically. If the coefficients of the subdiagrams are known to be $C_x(1)$ and $C_y(1)$, then

$$C_{xy} = [D_x C_x(1)] * [D_y C_y(1)] * (C_2^4)^2 T\,. \tag{47}$$

Here, the coefficient D reflects the weight of the subgraph channel. Figure 6(x) has a weight $D = 1/3$ in the horizontal channel, and $D = 2/3$ in the vertical channel. Figure 6(y) has weight $D = 2/3$ in the vertical channel, but in the horizontal channel due to left–right asymmetry each parquet has weight 1/6. A fully symmetric diagram has weight $D = 1$. If the original diagram has left–right symmetry then $T = 1$, otherwise $T = 2$. The factor $C_2^4 = 6$ is the number of combinations when picking up two external lines in a four-point subgraph. Take, for example, Fig. 3(i), it is the result of combining two Fig. 3(a), therefore

$$C_i(1) = \left[\frac{1}{3}C_a(1)\right] * \left[\frac{2}{3}C_a(1)\right] \times 72 = 20736\,.$$

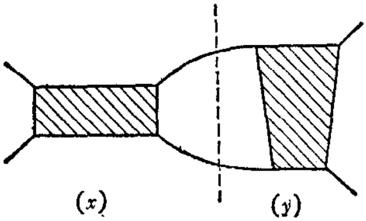

Fig. 6.

When $n > 1$, we first consider a case with fixed spin indices. The coefficient of the original graph is still given by Eq. (47), but when changing $C(1)$ to $C(n)$ in Eq. (46), $P(n)$ must be taken as that before the symmetrization. Only the final result is subject to symmetrization. When the weight in the vertical channel is $D = 2/3$, one may write down explicitly the two combinations of the spin indices to be summed over and use $D = 1/3$ instead. Take again Fig. 3(i) as example,

$$\left[\frac{1}{3}C_a(1)\frac{\Delta_2(\alpha\beta;\mu\nu)}{9}\right]\left[\frac{1}{3}C_a(1)\frac{\Delta_2(\mu\gamma;\nu\delta)+\Delta_2(\mu\delta;\nu\gamma)}{9}\right]\times 72 = 256\omega_4(\alpha\beta;\gamma\delta)\,,$$

yielding $C_i(n) = 256(3n^2 + 22n + 56)$ after symmetrization. Since the definition of vertex part includes the symmetric tensor and the factor $1/3$, calculation of the coefficients of skeleton graphs reduces to that of the perturbation diagram coefficient $C(1)$. For example, the coefficients in Eq. (24) are $C_a(1) = 36$ and $C_c(1) = 432$.

7. Some Integrals

We list the integrals used in this work. These integrals are calculated in a d-dimensional spherical coordinate system, expanded to the required power of ϵ. In diverging integrals we introduce a momentum cut-off Λ and retain the nonvanishing terms at $\Lambda \to 0$. The subscripts of the following integrals are the same as the labels in the corresponding diagrams. Usually the derivatives of some integrals are easier to calculate than the integrals themselves. In the following formulas, we have $L = \Lambda/\sqrt{r}$, the Euler constant $C = 0.577216$, the Riemann ζ function $\zeta(3) = 1.202057$, and an integral

$$C_1 = \int_1^\infty \frac{\ln x}{x^2 - x + 1}\frac{dx}{x+1} = 1.171954.$$

$$\begin{aligned} I(p,r) &\equiv I_a(p,r) \\ &\equiv \int G(q)G(p+q) = \frac{1}{(4\pi)^2}\left[1 + 2\ln L + \frac{\sqrt{p^2+4r}}{p}\ln\frac{\sqrt{p^2+4r}-p}{\sqrt{p^2+4r}+p}\right]\,, \end{aligned}$$

$$I \equiv I_a(0,r) = \frac{1}{(4\pi)^2}\left\{2\ln L - 1 + \frac{\epsilon}{2}\left[(2\ln L - 1)(1 - C + \ln\frac{4\pi}{r}) + \frac{\pi^2}{6} - 2(\ln L)^2\right]\right\}, \tag{48}$$

$$I(p) \equiv I_a(p,0) = \frac{1}{(4\pi)^2}\left\{(2\ln\frac{\Lambda}{p} + 1)[1 + \frac{\epsilon}{2}(\ln 4\pi - C)] + \epsilon\left[\frac{3}{2} - 2\ln p + \ln\Lambda + (\ln p)^2 - (\ln\Lambda)^2\right]\right\},$$

$$I_b(p,r) \equiv \int_q G\left(q - \frac{p}{2}\right) G\left(q + \frac{p}{2}\right) I(q,r),$$

$$I_b \equiv I_b(0,r) = -\frac{1}{(4\pi)^4}[1 + C_1 - 2(\ln L)^2], \tag{49}$$

$$I_b' \equiv \frac{d}{dr} I_b(0,r) = \frac{1}{(4\pi)^4 r}\left\{-2\ln L + \epsilon\left[C_1 - \frac{\pi^2}{12} + 2\left(C - \ln\frac{4\pi}{r} - \frac{1}{2}\right)\ln L + (\ln L)^2\right]\right\},$$

$$I_e' = \frac{1}{(4\pi)^6 r}[1 + 2C_1 - 4(\ln L)^2], \tag{50}$$

$$I_f' = \frac{1}{(4\pi)^6 r}[C_1 - 2\ln L - 2(\ln L)^2], \tag{51}$$

$$I_g' = \frac{1}{(4\pi)^6 r}[2 - 4(\ln L)^2], \tag{52}$$

$$I_h' = -\frac{6}{(4\pi)^6 r}\zeta(3), \tag{53}$$

$$I_i \equiv \int G^3(k) I(q,r)[G(q+k) - G(q)], \quad I_f' = \frac{1}{(4\pi)^6 r}\left(\frac{1}{2} + \ln L\right). \tag{54}$$

8. Generalized Homogeneous Functions and the Callan–Symanzik Equation

Consider a general function of, say, three variable $f(x_1, x_2, x_3)$. If under a scale change in all dimensions $x_i \to \lambda x_i$, the function remains the same except for multiplying by a numerical factor λ^n:

$$f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^n f(x_1, x_2, x_3), \tag{55}$$

then f is a homogeneous function of order n. Differentiating both sides of Eq. (55) with respect to λ and letting $\lambda = 1$, we get a partial differential equation, namely, the Euler equation for homogeneous functions:

$$\left[x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right] f(x_1, x_2, x_3) = nf(x_1, x_2, x_3)\,. \tag{56}$$

The expression in square brackets $\sum_i x_i\frac{\partial}{\partial x_i}$ is called a dilation operator.

If the scale change is performed differently in different dimensions $x_1 \to \lambda^{\alpha_1}x_1$, $x_2 \to \lambda^{\alpha_2}x_2$, $x_3 \to \lambda^{\alpha_3}x_3$ and the function remains the same up to a common factor λ^n:

$$f(\lambda^{\alpha_1}x_1, \lambda^{\alpha_2}x_2, \lambda^{\alpha_3}x_3) = \lambda^n f(x_1, x_2, x_3)\,, \tag{57}$$

then f is a generalized homogeneous function. It satisfies a generalized Euler equation

$$\left[\alpha_1 x_1\frac{\partial}{\partial x_1} + \alpha_2 x_2\frac{\partial}{\partial x_2} + \alpha_3 x_3\frac{\partial}{\partial x_3}\right] f(x_1, x_2, x_3) = \lambda^n f(x_1, x_2, x_3)\,. \tag{58}$$

The dilation operator becomes

$$\alpha_1 x_1\frac{\partial}{\partial x_1} + \alpha_2 x_2\frac{\partial}{\partial x_2} + \alpha_3 x_3\frac{\partial}{\partial x_3}\,.$$

Many scaling relations may be derived if one assumes that thermodynamic functions near critical points are generalized homogeneous functions.[15]

If, in addition, the function contains a parameter R which depends on the factor λ during the scale change, i.e., $R \to R(\lambda)$. Then there appears a term in the generalized Euler equation as well as in the dilation operator:

$$\left[\sum_i \alpha_i x_i\frac{\partial}{\partial x_i} + \beta\frac{\partial}{\partial R}\right] f(x_1, x_2, x_3) = \lambda^n f(x_1, x_2, x_3)\,, \tag{59}$$

where a coefficient β is introduced:

$$\beta = \left.\frac{dR}{d\lambda}\right|_{\lambda\to 0}\,. \tag{60}$$

This is an elementary derivation of the Callan–Symanzik equation without making use of any knowledge of field theory. At critical point, the correlation length diverges and the scale change does not affect the generalized function at all. In other words, at the critical point $\beta = 0$. The critical exponents may be calculated from the zero of the coefficient function β in the Callan–Symansik equation. Had we been aware of this elementary derivation of Callan–Symansik equation 40 years ago we could have explained the relation of our skeleton graph expansion to other field theory calculation of critical exponents then.

Note: In the Reference list, we included only those available when writing paper of Ref. 9 except for Ref. 1.

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