UNIVERSAL SLOWING-DOWN EXPONENT NEAR PERIOD-DOUBLING BIFURCATION POINTS

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The critical exponent Δ, describing the slowing of convergence near period-doubling bifurcations, should be a universal constant with "mean-field" value $\Delta = 1$.

In connexion with the recent discovery of universal properties in period-doubling bifurcations [1–4] and their comparison with scaling and universality in phase transitions [5] we would like to point out that there exists a universal exponent, related to the "critical dynamics" of these bifurcations. In analogue with critical slowing down in phase transitions, the time constant describing the convergence near the kth bifurcation point λ_k behaves like

$$\tau \propto |\lambda - \lambda_k|^{-\Delta} , \tag{1}$$

but this is only an one-sided phenomenon, observable when approaching λ_k from the less-bifurcated state.

Of course, slowing down itself has been a well known fact, since in numerical practice the worsening of convergence has long been considered as a symptom of a nearby instability in the algorithm used. What might be new is the existence of a universal characteristic for such phenomena.

Let us consider a nonlinear mapping [1] $x_{n+1} = \lambda f(x_n)$ and denote the pth iterate of this function by

$$F(\lambda, p, x) \equiv (\lambda f)^p \circ (x) ,$$

where $p = m 2^k$ with fixed m in between λ_k and λ_{k+1} in a bifurcation sequence. Without loss of generality we assume that bifurcation occurs when λ crosses λ_k from below. Then, given p (or k), the mapping $x_{n+1} = F(\lambda, p, x_n)$ converges to certain fixed point $\bar{x}$ for $\lambda < \lambda_k$, including all λ_i with $i < k$. Let $x_n = \bar{x} + \epsilon_n$, we have as usual

$$\epsilon_{n+1} = F'(\lambda, p, \bar{x})\, \epsilon_n \tag{2}$$

near λ_k. The convergence condition $|F'| < 1$ breaks down at λ_k, where

$$|F'(\lambda_k, p, \bar{x})| = 1 . \tag{3}$$

Assume that $|\epsilon_n|$ diminishes as $\exp(-n/\tau)$, then it follows immediately from (2) that

$$\tau = -(\ln|F'(\lambda, p, \bar{x})|)^{-1} = -\left(p \ln\lambda + \sum_{i=1}^{p} \ln f'(\bar{x}_i)\right)^{-1} ,$$

where $\bar{x}_i$ are points forming the p-cycle. Expanding $\ln\lambda$ near λ_k, we get due to (3)

$$\tau = \lambda_k / p(\lambda_k - \lambda) , \tag{4}$$

i.e. eq. (1) with $\Delta = 1$. It coincides with the mean-field theory value of Δ in the conventional theory of critical dynamics (for a recent review see [6]).

It would be much more interesting to look for deviations, presumably still universal, of Δ from 1 and look for nonexponential decay of $|\epsilon_n|$ in the vicinity of λ_k. Our preliminary numerical check on a few iterated maps led to a value for Δ very close to 1. Study on systems, described by differential equations, is in progress.

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