SUBHARMONIC STROBOSCOPY AS A METHOD TO STUDY PERIOD-DOUBLING BIFURCATIONS

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We suggest to use subharmonic stroboscopic sampling to resolve high-order bifurcations in a numerical study of driven nonlinear oscillators. A bifurcation sequence up to the 1024th period has been located in the forced Brusselator by using this method.

Period-doubling bifurcations, leading to chaotic motion, have been observed in many nonlinear systems, driven by an external periodic force [1–4]. In contrast to nonlinear mappings, i.e. difference equations, where bifurcations of rather high orders have been identified with remarkable precision [5–7], it is much more difficult to resolve them in systems described by differential equations. With a 8192 points FFT (fast Fourier transform), one can hardly go beyond the 64th subharmonic without serious aliasing. The situation becomes even more difficult when one tries to distinguish chaotic bands and periodicities embedded in them.

We have been using a simple extension of the usual stroboscopic sampling idea [8,9], i.e. to sample also at the subharmonic frequencies instead of sampling at the fundamental external frequency only. As an illustration, we report a few results on the Brusselator [10] with periodic force added:

$$\dot{X} = A - (B+1)X + X^2 Y + \mathrm{d}\cos(\omega t)\,, \qquad \dot{Y} = BX - X^2 Y\,. \tag{1}$$

This system has been studied by Tomita and Kai [1,9] but their work was completed before the upsurge of papers, triggered by the discovery of universal properties in nonlinear mappings [5]. Therefore, it is interesting to look at it with higher resolution.

Fig. 1 shows the process of identifying a $512T$ period, where $T = 2\pi/\omega$. Fig. 1a corresponds to the usual stroboscopic portrait, i.e. sampling X and Y at interval T. Fig. 1b was sampled at interval $8T$ and we thus amplified one of the 8 islands seen in fig. 1a. Further increase of the sampling period to $64T$ gave 8 distinct points. The systematic shift of samples for each point was caused by accumulation of truncation errors and could be reduced by using smaller integration steps and working in double precision.

By combined use of subharmonic stroboscopy and FFT analysis we have successfully located a few rather long period-doubling sequences in (1); an estimate of the ratio

$$\delta_n = (\lambda_n - \lambda_{n+1})/(\lambda_{n+1} - \lambda_{n+2})\,, \tag{2}$$

where λ_n denotes ω_n or α_n depending on which of ω and α is used as control parameter, shows that most probably they are Feigenbaum [5] sequences. In table 1, we give an example. To have more comparable date the parameter values were taken to be the same as in ref. [9], i.e. $A = 0.4$, $B = 1.2$ and $\alpha = 0.05$.

Being a kind of time series sampling, subharmonic stroboscopy suffers from the same demerits as FFT, i.e. nonuniqueness in interpretation and unability to tell frequencies, higher than the sampling frequency. If the actual period T_0 of the oscillator is in

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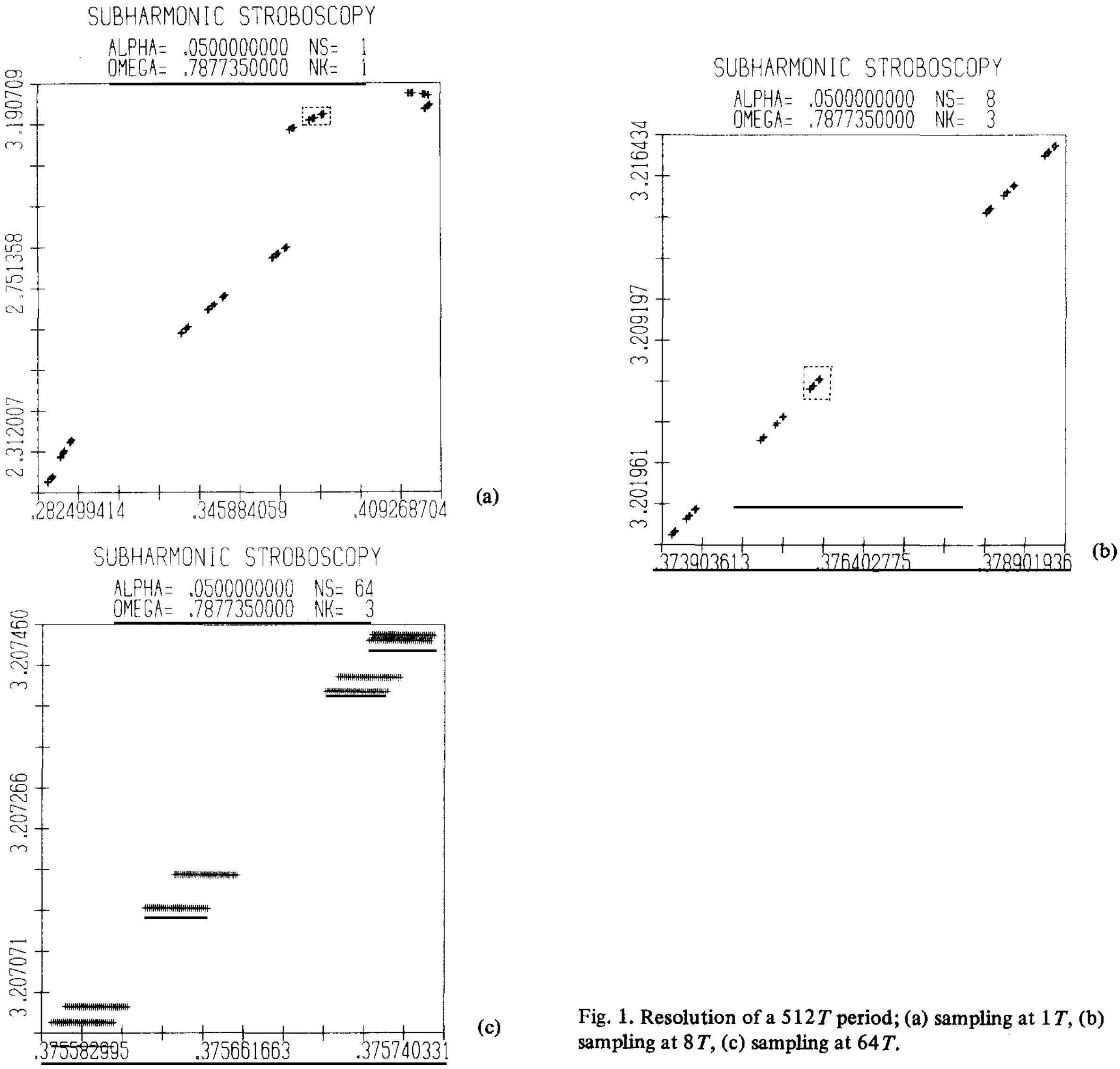

Fig. 1. Resolution of a $512T$ period; (a) sampling at $1T$, (b) sampling at $8T$, (c) sampling at $64T$.

Table 1
A period-doubling bifurcation sequence in (1).

n	Number of periods	Range in ω	ω_n	δ_n
1	1	0.398 20	0.398 205	5.53
2	2	0.398 21 – 0.713 05	0.713 062 5	4.24
3	4	0.713 075 – 0.769 996	0.769 999 8	4.02
4	8	0.770 00 – 0.783 37	0.783 435	4.46
5	16	0.783 50 – 0.786 752	0.786 776	4.41
6	32	0.786 80 – 0.787 52	0.787 525	5.40
7	64	0.787 53 – 0.787 69	0.787 695	4.04
8	128	0.787 70 – 0.787 726	0.787 726 5	4.88
9	256	0.787 727 – 0.787 734	0.787 734 25	
10	512	0.787 734 5 – 0.787 735 9	0.787 735 95	
11	1024	0.787 736		

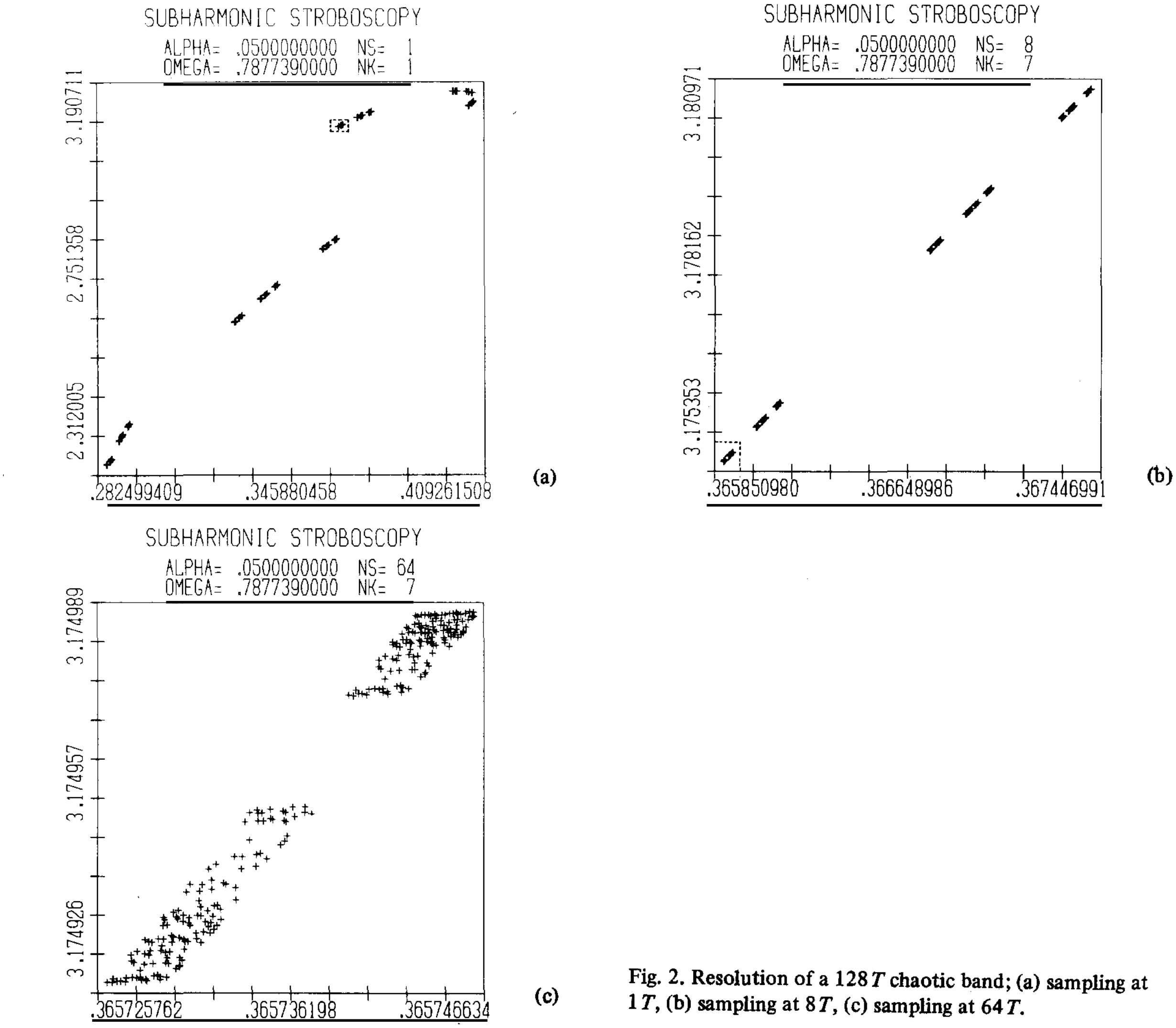

Fig. 2. Resolution of a 128 T chaotic band; (a) sampling at 1 T, (b) sampling at 8 T, (c) sampling at 64 T.

rational relation to the period T of external force

$$T_0 = (n/m)T\,, \tag{3}$$

m being incommensurable with n (subharmonic entrainment), the stroboscopic portrait at interval T gives n points for any $m \geqslant 1$. If $n = pq$, one can sample at interval pT or qT and increase the resolution thereby, but misuse of p, which is not a factor of n, will add a spurious multiplier p in the number of periods. To be safe, one should always go from low order subharmonics to higher ones and compare the results with FFT analysis. From the practical point of view the resolution power of subharmonic stroboscopy is limited only by computer time while the FFT encounters also storage limitations.

This method appears to be very useful in studying the chaotic bands in the inverse bifurcation sequence. Fig. 2a can hardly be distinguished from fig. 1a, but sampling at 64T, i.e. fig. 2c, shows a qualitatively different picture as compared with fig. 1c. This is a period 128 chaotic band in the inverse sequence just beyond the accumulation point of the direct sequence shown in table 1.

Using subharmonic stroboscopic sampling, supplemented by extensive power spectra analysis, we have studied the structure of chaotic bands and the systematics of periodicities embedded in them. A detailed account will appear elsewhere.

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