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ELEMENTARY SYMBOLIC DYNAMICS

and Chaos in Dissipative Systems

Hao Bai-lin
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and Chaos in Dissipative Systems

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Preface

Perhaps the first question to be touched on in this foreword is why write yet another book on chaos, since many monographs have appeared in recent years (see, e.g., Part I of the References at the end of this book, where many monographs, conference proceedings and collections of papers are listed). While working on the manuscript I have skimmed through the available literature with the hope that someone else would have released me from the task. However, as always I could not get rid of the thought that there exist some gaps in the literature on chaos. Many monographs authored by pure or applied mathematicians were competently written, but to a practitioner in physical sciences, he often feels the "language barrier". A few excellent books were written by our fellow physicists, but most were on an introductory level. Many topics such as the renormalization group approach, the role of external noise, transient behaviour and characterization of the attractors in laboratory and computer experiments, and especially, symbolic dynamics as a practical tool in studying chaos, have not been exposed in sufficient detail.

This book, however, is still an introduction to the study of chaotic behaviour in simple mathematical models, written by a physicist for those working in the physical sciences. The models considered are well-known ones, ranging from one-dimensional mappings to systems of a few ordinary differential equations. Throughout the book emphasis is laid on the use of symbolic dynamics in an elementary way, but we touch on a few other methods to study such models. Symbolic dynamics itself has been a long-studied mathematical topic and is still largely wrapped in fairly abstract (at least, for non-mathematicians) form. Most of the experts would agree that for the time being symbolic dynamics

might be the only rigorous way to define chaos and perhaps everyone should start the study of chaos from learning symbolic dynamics, but few practitioners can carry out this programme properly. We ourselves are learners. What we have been doing is to make use of a small part of this beautiful theory in a down-to-earth manner and we are glad to share the experience with colleagues from other branches of physical science.

The reader may find this book written quite inhomogeneously. Some parts contain only qualitative discussions, while other parts are furnished with detailed derivations. The author tried to use elementary mathematics and calculus, and, whenever possible, to rely on physical intuition. Substantial attention has been paid to numerical techniques in studying chaos, but the use of algebraic manipulation languages is also mentioned. I should admit that it is our intention to present the materials in this way in order to put those topics that deserve more scrutiny in depth on a more general background.

This book would not be possible without the interactions and communication with colleagues at home and abroad. There are too many names to be listed. At least, I would like to express my gratitude to the following scientists: P. Bak, T. Bohr, E. Brézin, Chen Shi-gang, R. Conte, P. Cvitanovic, Vic. Dotzenko, M. J. Feigenbaum, J. Ford, L. Glass, Gu Yan, G. Gunaratne, H. Haken, B. Hasslacher, Bambi Hu, Hu Gang, D. K. Kondepudi, C. Y. Liaw, P. Limcharoen, Liu Ji-xing, A. J. Mandell, G. Mayer-Kress, Ni Wan-sun, G. Nicolis, E. Ott, G. Parisi, Peng Shou-li, I. Percival, Y. Pomeau, I. Prigogine, L. Reichl, G. Schmidt, Ya. G. Sinai, H. E. Stanley, B. H. Stewart, H. L. Swinney, C. Tsallis, M. G. Velarde, Wang Guang-rui, Wang Xiao-jing, K. Wiesenfeld, K. M. Khanin, Xu Jing-hua, J. A. Yorke, K. Young, J. M. Yuan, Zhang Hong-jun, and Zheng Wei-mou, for discussions or hospitality or both. I would like to thank Drs. A. Arneodo, J. Bélair, Ping Chen, J. D. Farmer, G. Casati, C. Grebogi, He Da-ren, B. Hess, B. A. Huberman, J. L. Hudson, M. Inoue, K. Kaneko, H. Mori, K. Nakamura, V. I. Oseledes, R. E. Rapp, G. Riela, D. Ruelle, T. Tél, C. Tresser, K. Tomita, Y. Ueda, F. Vivaldi, and many others for providing their publications.

I would like to emphasize that discussions with Dr. Zheng Wei-mou, who has returned to our team recently, have deepened my understanding of symbolic dynamics and changed significantly the presentation of Chapters 3 and 4. Still

I have not been able to take into account all his comments, since a cut-off point must be introduced otherwise the manuscript would never be finished. In addition, I must mention my students Zeng Wan-zhen, Ding Ming-zhou, Li Jia-nan, Lu Li-sha, Zou Chuan-ming, and Yang Wei-ming for I have certainly learnt more from them than what I could offer.

Special thanks go to Zhang Shu-yu who has taken the trouble with all the logistics of my research, from the implementation of algebraic manipulation languages and TEX, to assistance in numerical calculations.

About the reference convention in this book, references to the References at the end of the book are given as, e.g., Poincaré (B1899), the capital B indicating the first part of the References on "Books, Conference Proceedings and Collections of Papers", or Feigenbaum (1980a), addressing a paper in the second part of the References. A few citations to papers not included in the References are given in the footnotes. No efforts have been made to clarify the chronology of one or other statements. In a rapidly expanding and interdisciplinary field like *Chaos* there must have been many rediscoveries of important facts. It is better to leave these to the historians of science.

During the writing of this book I have been constantly perturbed by the limitation of using a foreign language. I apologize for any inconvenience caused by my broken English, although I have honestly tried to meet the definition of the latter, as given by H. B. G. Casimir[1]. Prof. Rainer Radok kindly read the manuscript and corrected the English. Nevertheless, it is my sole responsibility for all the mistakes and inexactitudes that still remain in the text.

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The text was typeset by the author using LATEX of Leslie Lamport with indispensable help from the staff of World Scientific. In particular, I would like to thank Dr. K. K. Phua, the Editor-in-Chief, and Misses P. H. Tham and K. Tan, the editors, for their patience and advice.

[1]H. B. G. Casimir, *Scientific American*, **194**(1956) 96.

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Chapter 1

Mathematical Models Exhibiting Chaos

Throughout this book we shall use the term "physics" in a broad sense, understanding it as a synonym for the physical sciences. In this chapter a few physical systems will be analyzed and modelled by nonlinear equations which may exhibit chaotic behaviour in a suitable range of the control parameters. We shall use some of these models in subsequent chapters to show one or other aspects of a mathematical theory or a certain numerical method, and to gain deeper insight into the phenomenon of chaos. We start with a question which has been asked by everyone who comes across the somewhat fuzzy word *chaos* in the context of an exact science like physics.

1.1 What Is Chaos?

We should consider the question at least in a threefold way: the phenomenon, the word, and the science. To begin with the phenomenon, let us recall an old experiment that has been redone in view of the new understanding of chaos.

In 1831, Faraday observed shallow water waves in a container vibrating vertically with a frequency ω and discovered the occurrence of a subharmonic component of frequency $\omega/2$. Later Lord Rayleigh repeated and discussed this

experiment from the viewpoint of parametric resonance in his famous treatise *The Theory of Sound* (first edition, 1877). Why did the appearance of subharmonics deserve special attention?

Many experimental apparatus in physics may be viewed as frequency transformers. A system is linear if the frequency components of the input and output signals coincide, and nonlinear otherwise. In nonlinear systems, higher harmonics as well as sums and differences of the input frequencies appear naturally, but it is not trivial to have subharmonics. In fact, if the output quantity Q_{out} depends on the input Q_{in} nonlinearly, e.g.,

$$Q_{out} = a\,Q_{in} + b\,Q_{in}^2$$

and Q_{in} contains a single frequency component, say, $Q_{in} \propto \cos(\omega t)$, then Q_{out} will contain frequency components 2ω, ω, and 0. If Q_{in} is a linear combination of $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$, then one can observe in Q_{out} also $\omega_1 \pm \omega_2$. All these are a simple consequence of basic relations among trigonometric functions. Furthermore, the appearance of frequency harmonics, sums and differences does not exhibit a threshold. They present themselves inevitably, perhaps, with very small amplitude, no matter how weak the nonlinearity may be. The subharmonic observed by Faraday and Rayleigh, however, was a kind of threshold phenomenon: it appeared suddenly when the nonlinearity reached a certain magnitude. The appearance of both the threshold and the subharmonic calls for a highly nontrivial explanation.

In 1981, the Faraday experiment was repeated with modern data acquisition and analysing systems (Keolian *et al.*, 1981; it contains also a few historical references). Not only the second subharmonic, but also a sequence of subharmonics such as 1/2, 1/4, 1/12, 1/14, 1/16,..., each with its own threshold, were found. This sequence then turned into a noise-like output with a continuous frequency spectrum. This is just what today people call chaos. 150 years after Faraday, scientists returned to his experiment, because it had become clear then that subharmonics usually appear as the first tone in the overture to chaos. In fact, in many nonlinear systems, various chaotic states of motion may be reached via a series of finite or infinite numbers of sharp transitions (sometimes referred to as scenario or route to chaos) and the final chaotic states are characterized by a number of quantities that distinguish them from pure randomness.

Broadly speaking, to classify various types of time evolution, one may think of the following possibilities:

1. Purely random processes, e.g., coin tossing.

2. Entirely deterministic processes, e.g., the two body problem in classical mechanics.

3. Deterministic processes subject to random fluctuations, e.g., Brownian motion of a pollen particle in water.

4. Seemingly random behaviour in deterministic systems without any external stochastic source. It is this category of phenomena that now comes under the term chaos.

5. Chaotic processes subject to external noise. Any experimental study of chaos, in laboratories or on computers, must deal with the inevitable effect of external noise and one must be able to distinguish chaos from noise.

It is curious enough that chaotic phenomena have been overlooked for centuries. Once they had been recognized, people began to see chaos everywhere, in Nature and in laboratories. Indeed, chaotic phenomena are ubiquitous whereas purely random or deterministic processes occur only as exceptions. The great time lag in recognizing chaotic phenomena might be attributed partially to the dominance of the deterministic point of view in science ever since the time of Newton, partially, until the 1960s, to the lack of means to tackle and visualize the complicated dynamics, i.e., the modern computers with their graphics.

The word *chaos* appeared as a scientific term in L. Boltzmann's assumption on molecular chaos in his derivation of the famous H-theorem more than a hundred years ago. N. Wiener used the word chaos in the titles of several papers[1]. Both scientists, however, used it to denote disorder caused by or closely related to stochastic processes. The modern usage of the word for intrinsic stochastic behavour in deterministic systems seems to have appeared

[1]N. Wiener, "The homogeneous chaos" (1938) and "The discrete chaos" (1943), both reprinted in *Collected Works*, ed. by P. Masani, MIT Press, 1976. We thank Dr. P. Limcharoen for telling us these references.

for the first time in the title "Period 3 implies chaos" of Li and Yorke's 1975 paper.

For the time being, there is still no generally accepted definition of chaos. Indeed, there have been some nice and rigorous mathematical definitions for stochastic behaviour in dynamical systems, however, usually it is very difficult to fit a realistic system or model into the mathematical framework. Therefore, one tends to use a working or operational definition for chaos. If seemingly random motion occurs in a system, without applying any external stochastic forces, and the individual output depends on the initial conditions sensitively, but, at the same time, some global characteristics, (e.g., a positive Lyapunov exponent or entropy, fractal attractor dimension, etc., see Chapter 6) turn out to be quite independent of the initial conditions, then one may well be dealing with chaos. Fortunately, physicists rarely disagree on what they observe as being chaos. We shall see that symbolic dynamics provides us with a rigorous way to define chaos. For the time being, however, it is better to look at concrete models instead of playing with definitions.

1.2 The Concept of Universality and the Role of Models

It is appropriate to precede the models by a brief discussion of the deep change of our attitude towards the role of mathematical models in physics that has taken place during the last two decades. In order to grasp the essence of a physical phenomenon, one has to put aside all the secondary factors and to construct simple, yet nontrivial, mathematical models. Since few models can be solved rigorously, it is often necessary to resort to further approximations. It is quite natural, however, to raise questions on the conclusions of a model study as to what are the peculiarities introduced by the particular model, and what are the artifacts introduced by the adopted approximation. For many years, most scientists have paid respect to rigorously solved models, at least for their mathematical beauty, but have viewed with scepticism approximate model studies.

Successes in the theory of phase transitions and critical phenomena have

infused new meaning into the notion of universality. Many natural phenomena and mathematical models are grouped into classes characterized by similar behaviour in their parameter dependence, especially, when these parameters are close to some critical values where abrupt change takes place. As a rule, a great number of degrees of freedom are required to describe the routine evolution of a more or less complex physical system. The sudden change of the system at a certain transition point, however, may be characterized by only a few variables. Generally speaking, it is sharp transition rather than smooth development that reveals the universal nature of systems. In fact, these remarks are backed by some weighty mathematical cornerstones which we shall not touch on in this book. Nevertheless, we shall study models consciously as representatives of relevant universality classes.

Another factor that has influenced our viewpoint on models is the use of high-speed digital computers. No matter how difficult a model may be, we are now able to simulate most of its behaviour on computers, provided, of course, necessary precautions are taken. The barrier between analytically solvable and unsolvable models has been diminishing since the introduction of computers. Nowadays, even analytical manipulation of models benefits from using computers. In this book, we shall pay due attention to both numerical and analytical aspects of the use of computers.

Historically, there have been a few models that have greatly stimulated the development of science. Over the decades, they have served as paradigms or touchstones for the development of many important theories. In the first place, we have in mind the two-body problem, starting from the classic Kepler problem of celestial motion, passing through the explanation of Mercurian perihelion procession in relativistic theory and hydrogen atom spectra in quantum mechanics, culminating in the impact of the understanding of the Lamb shift on the development of nonrelativistic and relativistic quantum field theory. As a second example, one may recall Brownian motion, which has been the source of inspiration for the whole stochastic approach in physical sciences, from Langevin and Fokker-Planck equations, path integral representation of Wiener to the Onsager-Machlup functional. In studying chaotic behaviour in nonlinear systems, we are lucky enough to have another such paradigm, namely, one-dimensional mappings of the interval, with which we start our study.

1.3 Insect Population and the Logistic Map

The simplest example of a nonlinear dynamical system comes from ecological models. Suppose there is a seasonally breeding insect population in which generations do not overlap. Then the average (or total, or maximal, depending on which quantity is measured) population Y_{n+1} of the next, i.e., $(n+1)$-th, generation will be determined entirely by the population Y_n of the present generation, i.e.,

$$Y_{n+1} = \Phi(Y_n).$$

This is a first order difference equation whose simplest possible form is a linear relation

$$Y_{n+1} = A\,Y_n. \tag{1.1}$$

The linear difference equation (1.1) can be easily solved to yield

$$Y_{n+1} = Y_0\,A^n,$$

which states that, if, on an average, each insect lays A eggs and all eggs hatch, then the population will grow exponentially, provided $A > 1$. If so, it would take only a few tens of generations before the globe would be overpopulated solely by this single species. However, new phenomena come into play when Y_{n+1} gets large enough: the insects will fight and kill each other for limited food, a contagious epidemic may sweep through the population, etc. Either fighting or touching requires the contact of at least two insects, and the number of such events is proportional to Y_{n+1}^2 (or, more pedantically, to $\frac{1}{2}Y_{n+1}(Y_{n+1}-1)$, that makes no difference when Y_{n+1} is large). Taking into account this suppressing factor, we can modify Eq. (1.1) into

$$Y_{n+1} = A\,Y_n - B\,{Y_n}^2. \tag{1.2}$$

Despite its apparently simple form, Equation (1.2) may exhibit a quite complicated dynamical behaviour, as we are going to learn in this book (see especially Chapter 2). Obviously, one of the two parameters A and B in (1.2) can be scaled out. Usually, one normalizes Y_{n+1} as well and writes (1.2) in one of the two following forms,

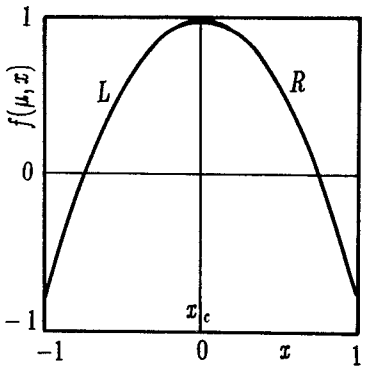


Figure 1.1: The logistic map.

$$y_{n+1} = 4\lambda y_n(1 - y_n), \quad y \in (0,1), \quad \lambda \in (0,1) \tag{1.3}$$

or

$$x_{n+1} = 1 - \mu x_n^2, \qquad x \in (-1,+1), \quad \mu \in (0,2). \tag{1.4}$$

Since both expressions appear frequently in the literature, we will write down the corresponding transformation for quick reference. In order to go from (1.3) to (1.4), let

$$y = (\lambda - 1/2)x + 1/2,$$
$$\mu = 4\lambda(\lambda - 1/2).$$

The reverse transformation reads

$$x = \frac{1-k}{\mu}y + \frac{k-1}{2\mu},$$
$$\lambda = (1-k)/4 \text{ where } \quad k = \pm\sqrt{1+4\mu}.$$

Note that the parameter range $\lambda \in (0,1)$ corresponds nonmonotonically to $\mu \in (-1/4, 2)$. This difference does not matter as long as we are concerned with chaotic regimes and the associated periodicities. Sometimes (1.4) is modified to read

$$x_{n+1} = \mu - x_{n+1}^2, \qquad x \in (-\mu, +\mu), \quad \mu \in (0,2). \tag{1.5}$$

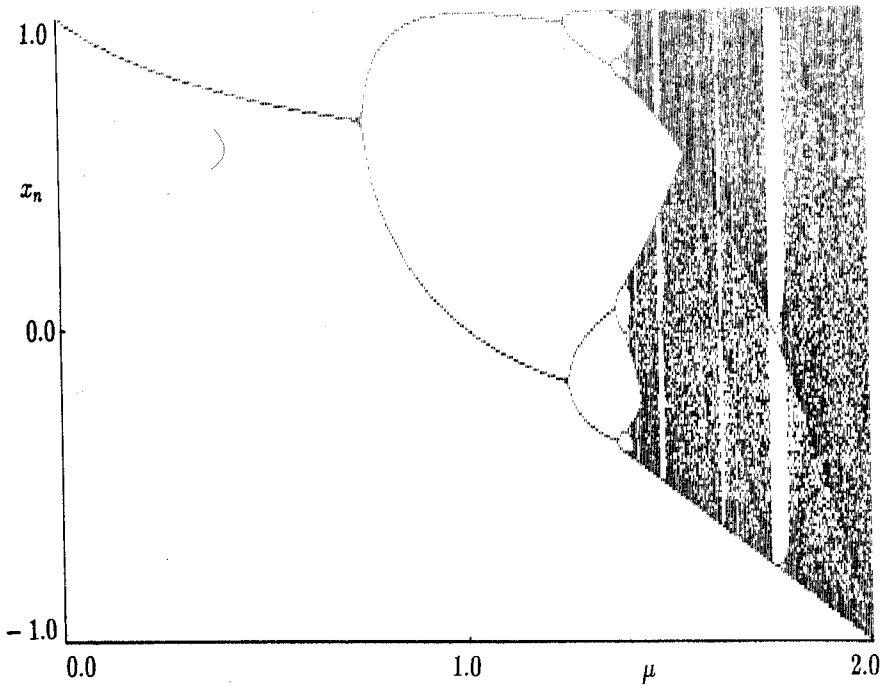


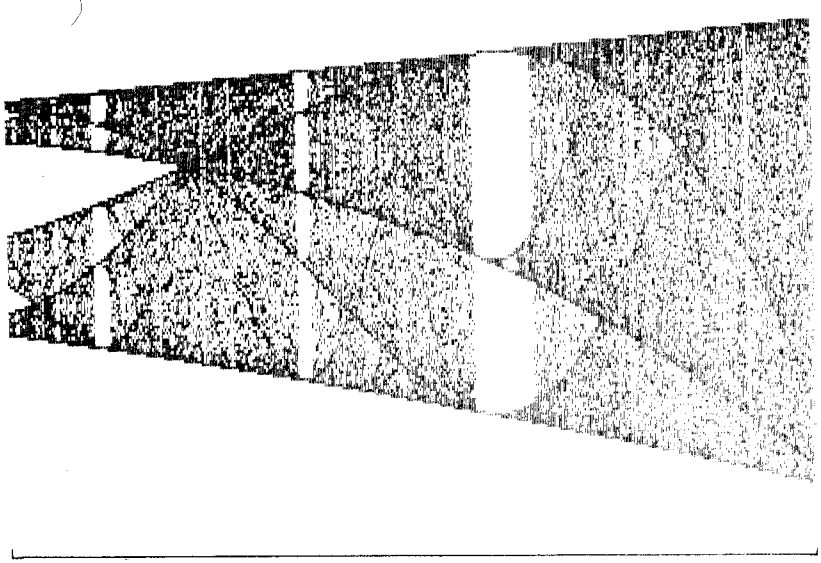
Figure 1.2: The bifurcation diagram of the logistic map.

Throughout this book we shall adhere to (1.4). This first order difference equation describes the time evolution of the normalized population x_n. Starting from a number x_n belonging to the interval $I = (-1, +1)$, it generates in a deterministic manner the next number x_{n+1} belonging to the same interval I, i.e., the nonlinear transformation $f(\mu, x) \equiv 1 - \mu x^2$ "maps" the interval I into itself. In the literature Equations (1.3) or (1.4) are often referred to as the *logistic map*. The function (1.4) is shown in Fig. 1.1. The letters R and L denote the *R*ight and *L*eft monotone branches of the map. These letters are the symbols which we shall be playing with throughout this book (see Chapter 3).

The simplest way to get a feeling regarding the logistic map is to visualize it on a personal computer with graphics. One covers the parameter range $\mu \in (0, 2)$ by small steps, and calculates, at each parameter value, the iterates of (1.4) starting from some initial value, say, $x_0 = 0.5$. After throwing away a few hundred points as transients (more about transient regimes in Chapter 7), one displays the remaining points on the screen. In this way we get the *bifurcation diagram* in $x - \mu$ coordinates (see Fig. 1.2). At every given parameter value, plotted along the x direction, one has the "limiting set" of the mapping. For $\mu \leq 0.75$, the limiting set consists of one point. This is the *fixed point* of the mapping. For $\mu = 0.75$ to 1.25 the limiting set comprises two points, giving rise to a 2-cycles or period 2 orbit. Then there come consecutively the 4, 8, ..., 2^n, ... cycles, forming a *period-doubling bifurcation sequence.* If one looks downwards along the μ axis, starting from $\mu = 2$, then one encounters first a one-band chaotic region, followed by two, four, ... and 2^n-band chaotic regions. Both sequences meet at a certain $\mu_\infty = 1.40115\ldots$. Within a chaotic region, there are many *periodic windows*, i.e., lucid intervals where only periodic orbits exist instead of chaotic output. Moreover, we see many dark lines, either going through chaotic regions or becoming boundaries of the latter.

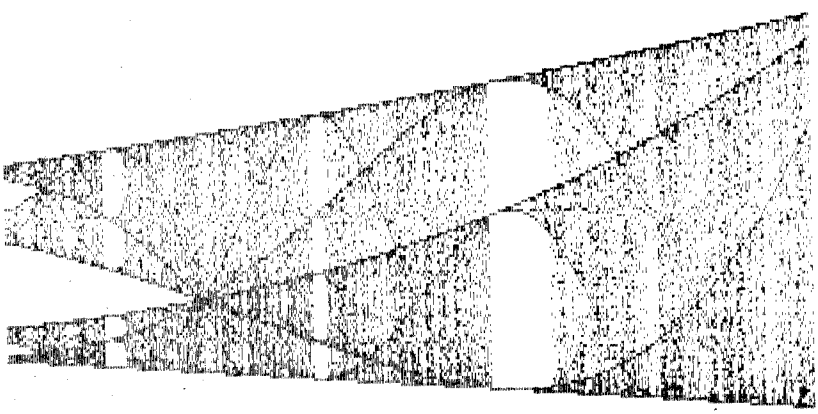
A bifurcation diagram is essentially a diagram of attractors, because almost all initial points are attracted to the points shown in the figure, provided a sufficient number of transients has been thrown away. Fixed points and periodic points are trivial attractors, while the darkened vertical segments are chaotic attractors.

One can "zoom" into the details of the bifurcation diagram by changing to smaller and smaller scales both in x and in μ. Figure 1.3 is a blow-up of a small part of Figure 1.2, namely, for $\mu = 1.78$ to 1.79 and $x_n = -0.16$ to 0.16. Figure 1.4 is a further blow-up of a small part of Figure 1.3, namely, for $\mu = 1.78632$ to 1.78650 and $x_n = -0.02$ to 0.02. This process can be repeated *ad infinitum*, and the self-similar structure is obvious. If all that we have described occurs only for some particular map, then we would be dealing with no more than a rare species in the nonlinear zoo. It is remarkable that all these features happen to be shared by many nonlinear systems. Both the global structure of the bifurcation diagram and the numerical characteristics of many local transitions within the diagram are universal properties. We shall



1.78 μ 1.79

Figure 1.3: Blow-up of a small part of the preceding Figure 1.2.



1.78632 μ 1.78650

Figure 1.4: Further blow-up of a small part of the preceding Figure 1.3.

study this bifurcation structure in detail in Chapters 2 and 3.

At this stage, we point out that the logistic map serves as a representative of a wide class of one-dimensional mappings, i.e., of functions with only one hump. Those universal properties that depend on there being only one local maximum, but not on the nature of the maximum, are sometimes called *structural universality* (Derrida, Gervois and Pomeau, 1978 and 1979). Close to the local maximum at $x = x_c$, if the nonlinear function f may be expanded in the form

$$f(x) = f_{max} - A\,(x - x_c)^z + \cdots,$$

those properties, which are shared by maps with one and the same value of z, are classified as *metric universality.* Since z values other than 2 require additional conditions to be satisfied, e.g., $z = 4$ necessitates simultaneous vanishing of the second and third derivatives of f at x_c, the $z = 2$ maps, like the logistic map, represent the most generic case. They are sometimes called *unimodal maps* because they have only one hump. The necessary conditions that these unimodal maps must satisfy in order to enjoy the universal properties, will be formulated below in Chapter 2. The apparent simplicity of these maps may be deceitful, since their dynamics appears to be so rich, that one cannot say even now that everything related to these maps has been fully understood. In fact, a substantial part of this book will be devoted to the scrutiny of these maps.

1.4 Thermal Convection, the Lorenz Model and the Antisymmetric Cubic Map

Many mathematical models have been devised to simulate the turbulent processes in the earth's atmosphere; obviously, this is a problem of great concern to mankind. A simplified model treats the problem as thermal convection of a fluid between two infinite plates, subject to a temperature gradient. If one or both of the horizontal boundaries are considered to be free surfaces, one arrives at the Bénard problem. If both, the upper and lower, interfaces obey rigid boundary conditions, the problem reduces to the case first studied by Lord Rayleigh. In the Boussinesq approximation, the two-dimensional Rayleigh-Bénard problem leads to a pair of partial differential equations (Saltzman, 1962):

$$\begin{aligned}\frac{\partial \triangle\psi}{\partial t} &= -[\frac{\partial\psi}{\partial x}\frac{\partial\triangle\psi}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\triangle\psi}{\partial x}] + \sigma\frac{\partial\theta}{\partial x} + \sigma\triangle^2\psi, \\ \frac{\partial\theta}{\partial t} &= -[\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x}] + R\frac{\partial\psi}{\partial x} + \triangle\theta,\end{aligned} \tag{1.6}$$

where ψ is the stream function, θ is the deviation of the temperature from the linear profile, established by thermal conduction only, and the dimensionless parameters σ and R are the Prandtl and Rayleigh numbers, respectively.

If the boundary conditions are taken to be periodic in the horizontal directions, and free in the vertical direction, one has

$$\begin{aligned}&\psi = \triangle\psi = \theta = 0, \qquad (z = 0,\ \pi), \\ &\psi = \triangle\psi = \frac{\partial\theta}{\partial x} = 0, \qquad (x = 0,\ \pi/a),\end{aligned}$$

where $1/a$ is the aspect ratio. In this form Equations (1.6) could only be solved by numerical calculation. Such a brute force approach, however, would keep us from gaining deeper insight into the mechanism by which the motion becomes more and more chaotic through a series of transitions. A better way to look into this type of physics involves truncation of the system and its transformation into a system of ordinary differential equations, describing the evolution of a finite number of "modes". In this manner, Saltzman (1962) and Lorenz (1963) arrived at their celebrated model of three ordinary differential equations. As hundreds of papers and a monograph (Sparrow, B1982) have been devoted to the Lorenz model, instead of going into details, we confine ourselves only to an outline of how the Lorenz system appears as the first one in an hierarchy of systems (Curry, 1978; Zhong and Yang, 1986).

A systematic way of truncation of a system of partial differential equations is the Galërlkin expansion that satisfies the boundary conditions from the outset. In our case, a choice of M sine modes and $M+1$ cosine modes in the horizontal directions and N sine modes in the vertical direction, leads to the expansions:

$$\begin{aligned}\psi(x,z,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N}\psi_{m,n}\sin(amx)\sin(nz), \\ \theta(x,z,t) &= \sum_{m=0}^{M}\sum_{n=1}^{N}\theta_{m,n}\cos(amx)\sin(nz).\end{aligned} \tag{1.7}$$

On substitution of the above result into (1.6), all spatial derivatives disappear and one obtains a system of ordinary differential equations for the coefficient