Closed time path Green's functions and critical dynamics

Guang-zhao Zhou, Zhao-bin Su, and Bai-lin Hao
Institute of Theoretical Physics, Academia Sinica, Beijing, China

Lu Yu*
Lyman Laboratory, Physics Department, Harvard University, Cambridge, Massachusetts 02138
(Received 30 October 1979)

The closed time path Green's function (CTPGF) formalism is applied to the critical dynamics. The related results for the CTPGF approach are briefly reviewed. Three different forms of CTPGF's are defined, transformations from one to another form and other useful computation rules are given. The path integral presentation of the generating functional for CTPGF's is used to derive the Ward-Takahashi identities under both linear and nonlinear transformations of field variables. The generalized Langevin equations for the order parameters and conserved variables are derived from the vertex functional on the closed time path. The proper form of the equations for the conserved variables, including automatically the mode coupling terms, is determined according to the Ward-Takahashi identities and the linear response theory. All existing dynamic models are reobtained by assuming the corresponding symmetry properties for the system. The effective action for the order parameters is deduced by averaging over the random external field. The Lagrangian formulation of the statistical field theory is obtained if the random field one-loop approximation and the second-order approximation of order-parameter fluctuations on different time branches are both taken. The various possibilities of improving the current theory of critical dynamics within the framework of CTPGF's are discussed. The problem of renormalization for the finite-temperature field theory is considered. The whole theoretical framework is also applicable to systems near the stationary states far from equilibrium, whenever there exists an analog of the potential function ("free energy").

I. INTRODUCTION

The closed time path Green's function (CTPGF) formalism, developed by Schwinger[1] and Keldysh,[2] has been applied to a number of problems.[3] As pointed out by Zhou and Su,[4–6] this technique is quite effective in investigating the nonequilibrium statistical field theory. They used this method to analyze the Goldstone mode in the steady state for nonequilibrium dissipative systems such as unimode lasers in the saturation region.[7] In this article we apply the CTPGF formalism to study systems near equilibrium phase transition point. The complete system of equations for critical dynamics, including automatically the mode coupling terms and the Lagrangian formulation of the field theory are derived in a unified way. This provides a microscopic justification for the semiphenomenological models in critical dynamics and indicates various possibilities for improving the existing theory.

In the vicinity of the phase transition point the long-wave fluctuations dominate. Since the corresponding correlation length is much greater than the thermal wavelength, the quantum effect is irrelevant. However, in the quasiparticle description, such "purely" classical field theory does not correspond to the Boltzmann limit, but approaches the "super-Bose" case, i.e., the quasiparticle distribution $n \propto T/\epsilon$ where T is the absolute temperature (with $\hbar = C = k_B = 1$) and ϵ is the energy for the elementary excitation. Such a statistical field theory (or fluctuation field theory) has very close analogy with the usual quantum field theory.

In our viewpoint, the CTPGF formalism is a natural theoretical framework for studying such statistical field theory. Assuming the equilibrium density matrix for CTPGF we obtain automatically the ordinary quantum field theory for the low-temperature limit ($T << \epsilon$) and the existing static critical phenomena theory for the high-temperature case ($T >> \epsilon$) (see Appendix A). If the high-temperature limit near equilibrium state at the critical point is taken, the complete system of equations to describe the critical dynamics follows naturally, as will be shown later in this paper. By introducing the "response fields," noncommutative with the basic fields, Martin, Siggia, and Rose[8] constructed a classical statistical field theory (the MSR field theory) in close analogy with quantum field theory. As will be shown below the structure of the MSR field theory becomes clearer in the framework of CTPGF's.

In Sec. II we briefly summarize the related results for the CTPGF's, some of which are believed to be new, while others are known or have been published elsewhere.[4–6] A more or less complete list of formulas is given for reference convenience and to make up for the deficiency that papers[4–6] were published only in Chinese. The perturbation theory and the generating functional formalism for CTPGF's are outlined. Three different forms of general multipoint CTPGF's are defined and the transformations from one to another are described. Some computation rules which greatly simplify the usually complicated calculations involved when using the CTPGF are derived. These seem to be highly desirable especially in view of the fact that the technical complexity is one of the causes why the CTPGF approach has not found applications as wide as it deserves. These algebraic identities are shown to be the consequences of some basic properties of the CTPGF, opening new perspectives not inherent in the ordinary Green's-function formalism. The related properties of the two-point functions are outlined. The Feynman path integral presentation for the generating functional of CTPGF's is used to deduce the Ward-Takahashi identities under both linear and nonlinear transformations of the fields.

In Sec. III a short account of the existing theory of critical dynamics is given. The generalized Langevin equation, the mode coupling, and the Lagrangian formulation of the classical field theory are briefly reviewed to facilitate the comparison with the results in subsequent sections.

In Sec. IV the generalized Langevin equation for macrovariables is derived from the equation satisfied by the generating functional for vertex functions in the CTPGF formalism by differentiating the micro- and macro-time scales of variation and averaging over the micro-time scale. In general form this is true for both order parameters and conserved variables. The essential point is to determine the transport coefficient matrix $\gamma^{-1}(t)$ connecting these quantities. The proper form of the equation for conserved variables can be deduced from the Ward-Takahashi identities and the linear-response theory. Comparing this form of equation with the general one yields two blocks of the $\gamma^{-1}(t)$ matrix, one of which couples the conserved variables together and the other couples the conserved variables with the order parameters. The other two blocks of $\gamma^{-1}(t)$, one of which connects the order parameters and the other one connects the order parameters with the conserved variables are determined through symmetry considerations. It is important to emphasize that mode coupling terms appear naturally in these equations. They are not "introduced from the outside," as in the existing theory. Applications of the general theory to particular dynamic models are outlined.

In Sec. V the path integral formulation for the CTPGF's is used to derive the effective action for order parameters. Through Fourier transformation of the path integral the generating functional in the random external fields is introduced. Averaging over random fields yields the effective action, the general properties of which are also discussed. The most plausible trajectories are described by the time dependent Ginzburg-Landau equations, i.e., generalized Langevin equations without random forces. Fluctuations around the most plausible trajectories are considered. There is a possible new way of describing fluctuations in the CTPGF approach arising from the fact that field operators may take different values on positive and negative time branches. It turns out that with the one loop approximation of random fields which is equivalent to the Gaussian averaging, and with second-order fluctuations on different time branches the existing Lagrangian formulation of the classical statistical field theory, i.e, the MSR theory reappears.

In Sec. VI we summarize the main results obtained and discuss the potential possibilities of the CTPGF approach with regard to improving the existing theory of critical dynamics.

In Appendix A the problem of renormalization in the finite-temperature field theory is discussed. It is emphasized that near the phase transition point the leading infrared divergence has to be separated before the ultraviolet renormalization may be carried out. The necessity of using noncommutative operators to describe the time evolution of classical field theory is also discussed.

In Appendix B a proof is given for two theorems of Sec. II dealing with transformations among different forms of CTPGF's. Further useful examples and some technical details are described.

Throughout this paper we deal mainly with the applications of the CTPGF approach to dynamic critical phenomena, but it is clear from the presentation that the whole theoretical framework is also applicable to systems near stationary states far from equilibrium, provided the long-wave fluctuations are dominant.

II. SUMMARY OF THE CTPGF FORMALISM

A. Definitions and generating functionals

For simplicity we shall consider only multicomponent Hermitian Bose fields $\varphi(x)$. Extension to more general cases is obvious. The Lagrangian density can be written as

$$\mathfrak{L}=\mathfrak{L}_0(\varphi)-V(\varphi)-\varphi(x)J(x) \quad , \tag{2.1}$$

where $J(x)$ is the external field.

CTPGF for $\varphi(x)$ is defined as

$$G_p(1\cdots n)=(-i)^{n-1}\operatorname{tr}[T_p(\hat{\varphi}(1)\cdots\hat{\varphi}(n)\hat{\rho}] \ , \tag{2.2}$$

where $\hat{\varphi}(i)$ and $\hat{\rho}$ are the field operators and density matrix in the Heisenberg representation, index p indicates a closed time path consisting of positive $(-\infty,+\infty)$ and negative $(+\infty,-\infty)$ branches. The time variable t can take values on either branch. T_p is the time-ordering operator along the closed time path.

The generating functional for the CTPGF's is defined as

$$Z(J(x))=\operatorname{tr}\left\{T_p\left[\exp\left(-i\int_p\hat{\varphi}(x)J(x)\right)\right]\hat{\rho}\right\} \ , \tag{2.3}$$

where the integral is taken over the closed time path and the integration variable d^4x is omitted. In Eq. (2.3) the external fields on the positive and negative branches $J(x+)$ and $J(x-)$ are assumed to be different.

Taking functional derivatives with respect to $J(x)$ we obtain from Eq. (2.3)

$$G_p(1\cdots n)=i\left.\frac{\delta^n Z(J(x))}{\delta J(1)\cdots\delta J(n)}\right|_{J=0} \ . \tag{2.4}$$

In the interaction representation the generating functional (2.3) can be rewritten as

$$Z(J(x)) =\operatorname{tr}\left\{T_p\left[\exp\left(-i\int_p[V(\varphi_I(x))+\varphi_I(x)J(x)]\right)\right]\hat{\rho}\right\} \ , \tag{2.5}$$

where $\varphi_I(x)$ satisfies the Euler equation for the free fields. The interaction term can be taken from behind the trace operator to obtain

$$Z(J(x))=\exp\left[-i\int_p V\left(i\frac{\delta}{\delta J(x)}\right)\right] \times\operatorname{tr}\left\{T_p\left[\exp\left(-i\int_p\varphi_I(x)J(x)\right)\right]\hat{\rho}\right\} \ . \tag{2.6}$$

It is easy to show by generalization of the Wick theorem that

$$T_p\left[\exp\left(-i\int_p\varphi_I(x)J(x)\right)\right] =Z_0(J(x)){:}\exp\left(-i\int_p\varphi_I(x)J(x)\right){:} \ , \tag{2.7}$$

where : : means the normal product and $Z_0(J(x))$ is the generating functional for the free field

$$Z_0(J)=\exp\left(-\frac{i}{2}\int\int_p J(x)\Delta_p(x-y)J(y)\right) \ , \tag{2.8}$$

Δ_p being a free propagator.

Substituting Eq. (2.7) into Eq. (2.6), we obtain

$$Z(J(x))=\exp\left[-i\int_p V\left(i\frac{\delta}{\delta J(x)}\right)\right]Z_0(J(x))N(J(x)) \tag{2.9}$$

with

$$N(J(x))=\operatorname{tr}\left[{:}\exp\left(-i\int_p\varphi_I(x)J(x)\right){:}\hat{\rho}\right] \tag{2.10}$$

as the correlation functional for the initial state. $N(J(x))$ can be expanded into a series of successive cumulants

$$N(J(x))=\exp[-iW_N(J(x))] \ , \tag{2.11}$$

$$W_N(J(x)) =\sum_{n=1}^{\infty}\frac{1}{n!}\int_p\cdots\int_p N(1\cdots n)J(1)\cdots J(n) \ , \tag{2.12}$$

where

$$N(1\cdots n)=(-i)^{n-1}\operatorname{tr}[{:}\varphi_I(1)\cdots\varphi_I(n){:}\hat{\rho}]_c \ . \tag{2.13}$$

It is worthwhile to mention that correlation functions give contributions only on the mass shell and that they have the same value on different time branches because there is no time ordering operator T_p in the definition (2.10).

The perturbation theory in the CTPGF approach has the same structure as in the ordinary quantum field theory with the exception that the time integral is taken over the closed path, so every Feynman diagram is decomposed into 2^n diagrams, where n is the number of vertices. The presence of the initial correlations $N(12\cdots)$ which vanish for the vacuum state constitutes another difference from the ordinary theory. In principle we can take into account all orders of initial correlations, but actually we shall limit ourselves to the second cumulants.

It can be shown quite generally that the counter terms for the usual quantum field theory alone are enough to remove all ultraviolet divergences for the CTPGF's under the reasonable assumptions concerning the initial correlations.[6] We shall not touch this question here, but it should be mentioned that near the phase transition point the infrared singularities have to be separated first, so that the ultraviolet renormalization for the CTPGF's in this case is different from that for the ordinary field theory (see Appendix A). The generating functional for the connected CTPGF's is defined as

$$W(J(x))=i\ln Z(J(x)) \ , \tag{2.14}$$

$$G_p^c(1\cdots n)=\left.\frac{\delta^n W}{\delta J(1)\cdots\delta J(n)}\right|_{J=0} =(-i)^{n-1}\langle T_p[\varphi(1)\cdots\varphi(n)]\rangle_c \ , \tag{2.15}$$

where $\langle\ \rangle_c$ stands for $\mathrm{tr}(\cdots \hat{\rho})$, with the connected parts to be taken only.

The normalization condition for the generating functional is

$$Z(J(x))|_{J_+(x)=J_-(x)=J(x)}=1 \ , \qquad (2.16)$$

$$W(J(x))|_{J_+(x)=J_-(x)=J(x)}=0 \ . \qquad (2.17)$$

It is essential to point out that this condition does not require $J(x)$ itself to vanish in contrast to the ordinary Green's function formalism. We shall frequently make use of this basic property below.

As in the usual field theory, we perform the Legendre transformation for the generating functional

$$\Gamma(\varphi_c(x)) = W(J(x)) - \int_p J(x)\varphi_c(x) \ , \qquad (2.18)$$

where

$$\varphi_c(x) = \delta W(J(x))/\delta J(x) \ . \qquad (2.19)$$

As a consequence of Eq. (2.17), it follows from $J_+(x)=J_-(x)$ that

$$\varphi_{c+}(x)=\varphi_{c-}(x) \ . \qquad (2.20)$$

From the definition (2.18) we have

$$\delta\Gamma(\varphi_c)/\delta\varphi_c(x) = -J(x) \ . \qquad (2.21)$$

This is the basic equation for the vertex functional, from which the generalized Langevin equation will be derived.

Taking the functional derivative of Eq. (2.19) with respect to $\varphi_c(x)$ and that of Eq. (2.21) with respect to $J(x)$, we obtain

$$\int_p G_p^c(x,y)\Gamma_p(y,z) = -\delta_p^4(x-z) \ ,$$
$$\int_p \Gamma_p(x,y) G_p^c(y,z) = -\delta_p^4(x-z) \ , \qquad (2.22)$$

where the two-point vertex function

$$\Gamma_p(12) \equiv \frac{\delta^2\Gamma(\varphi_c(x))}{\delta\varphi_c(1)\delta\varphi_c(2)} \ . \qquad (2.23)$$

Actually Eq. (2.22) is the Dyson equation for the CTPGF's, from which the kinetic equation for the distribution N and the energy spectrum and dissipation for quasiparticles can be deduced.[3–5] Here $\delta_p(x-y)$ is the δ function on the closed time path. It is defined for arbitrary functions along the closed time path that

$$\int_p f(y)\delta_p(y-x) = f(x) \ , \qquad (2.24)$$

where x can take values on either branch of time.

Up to now we have considered CTPGF's for the basic fields $\varphi(x)$, but all the above statements about φ can be repeated word for word for all the composite operators $Q(\varphi(x))$.

B. Transformations of three sets of CTPGF

In the CTPGF approach we have to deal with three different forms of functions:

(a) Functions on the closed time path $G_p(12\cdots n)$ with subscript p, which appear under the integrals over the closed time path and are used for a concise writing of formulas. (b) The tensor functions $\hat{G}(12\cdots n)$ with time arguments on positive or negative time branches which appear under the integrals over the single time axis $(-\infty,+\infty)$ and are used for constructing the perturbation theory [in what follows we shall specify them by the Greek subscripts $G_{\alpha\beta\cdots\rho}(12\cdots n)$ with $\alpha,\beta\cdots=\pm$, etc.]. (c) The retarded, advanced, and correlation functions, representing the physical quantities $\tilde{G}(12\cdots n)$ which will be denoted by the Latin subscripts $\tilde{G}_{ij\ldots}(12\cdots)$ with $_{ij\ldots}=1,2$. Either of tensors $\hat{G}$ and $\tilde{G}$ has 2^n components.

Some of the relationships among these functions were given before,[2–4] but our main point is to generalize them to the multipoint function case.

To start with the transformations we specify first our notation. The Pauli matrices are written as

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \ .$$

σ_3 will appear frequently together with $\hat{G}$ and σ_1 with $\tilde{G}$. The real orthogonal matrix

$$Q=\frac{1}{\sqrt{2}}(1-i\sigma_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \ ,$$
$$Q^T=Q^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \qquad (2.25)$$

is used for the transformations between $\hat{G}$ and $\tilde{G}$.

The multipoint step function Θ is defined as

$$\Theta(1,2,\cdots n)=\begin{cases}1, & \text{if } t_1>t_2\cdots>t_n \ ,\\ 0, & \text{otherwise} \ .\end{cases} \qquad (2.26)$$

It is the product of the two-point step functions

$$\Theta(1,2,\cdots n)=\Theta(1,2)\Theta(2,3)\cdots\Theta(n-1,n) \qquad (2.27)$$

and can be used to define the T product

$$T(\varphi(1)\varphi(2)\cdots\varphi(n)) = \sum_{\mathcal{P}_n}\Theta(p_1,p_2\cdots p_n)\varphi(p_1)\varphi(p_2)\cdots\varphi(p_n) \ , \qquad (2.28)$$

where summation goes over all possible permutations.

The step functions satisfy some relations such as the normalization condition

$$\sum_{\mathcal{P}_n}\Theta(p_1,p_2\cdots p_n)=1 \tag{2.29}$$

and the sum rule ($n>m$)

$$\Theta(1,2,\cdots m)=\sum_{P_n(1,2\cdots m)}\Theta(p_1,p_2\cdots p_n)\ , \tag{2.30}$$

where $P_n(1,2\cdots m)$ means p_n with 1 preceding 2, 2 preceding 3, etc.

Going from G_p to $\hat{G}$ we have only to assign separately the "+" or "−" subscript in accordance to the value of the time argument. Take two-point function for example we have

$$\hat{G}(12)=\begin{pmatrix} G_{++}(12) & G_{+-}(12)\\ G_{-+}(12) & G_{--}(12)\end{pmatrix}\equiv\begin{pmatrix} G_F & G_+\\ G_- & G_{\tilde{F}}\end{pmatrix} \tag{2.31}$$

with

$$\begin{aligned}
G_F(12)&=-i\langle T(\varphi(1)\varphi(2))\rangle=i\frac{\delta^2 Z(J)}{\delta J_+(1)\delta J_+(2)}\bigg|_{J=0}\ ,\\
G_+(12)&=-i\langle\varphi(2)\varphi(1)\rangle=i\frac{\delta^2 Z(J)}{\delta J_+(1)\delta J_-(2)}\bigg|_{J=0}\ ,\\
G_-(12)&=-i\langle\varphi(1)\varphi(2)\rangle=i\frac{\delta^2 Z(J)}{\delta J_-(1)\delta J_+(2)}\bigg|_{J=0}\ ,\\
G_{\tilde{F}}(12)&=-i\langle\tilde{T}(\varphi(1)\varphi(2))\rangle=i\frac{\delta^2 Z(J)}{\delta J_-(1)\delta J_-(2)}\bigg|_{J=0}\ ,
\end{aligned} \tag{2.32}$$

where $\tilde{T}$ is the inverse time ordering operator.

The transformation proposed by Keldysh[2] for two-point functions

$$\tilde{G}(12)=Q\hat{G}(12)Q^{-1} \tag{2.33}$$

if written in components

$$G_{ij}(12)=Q_{i\alpha}Q_{j\beta}G_{\alpha\beta}(12) \tag{2.34}$$

can be generalized directly to the multipoint case

$$G_{i_1i_2\cdots i_n}(12\cdots n)=2^{n/2-1}Q_{i_1\alpha_1}Q_{i_2\alpha_2}\cdots Q_{i_n\alpha_n}G_{\alpha_1\cdots\alpha_n}(12\cdots n)\ . \tag{2.35}$$

The inverse transformation is given by

$$G_{\alpha_1\alpha_2\cdots\alpha_n}(12\cdots n)=2^{1-n/2}Q^T_{\alpha_1i_1}Q^T_{\alpha_2i_2}\cdots Q^T_{\alpha_ni_n}G_{i_1i_2\cdots i_n}(12\cdots n)\ . \tag{2.36}$$

We shall see below that Eq. (2.35) contains all possible retarded, advanced, and correlation functions, associated directly with the physical quantities. The expediency of such choice of numerical coefficient becomes clearer somewhat later.

The specific features of the CTPGF's in the form of tensors can be characterized by two theorems, proof of which and more involved examples will be postponed to the Appendix B.

a. Theorem 1. The component of $\tilde{G}$ with all subscripts equal to 1 vanishes, i.e.,

$$\tilde{G}_{11\cdots1}(12\cdots n)=0\ . \tag{2.37}$$

As consequences of this theorem for one-point and two-point functions we have

$$G_1(x)=0,\quad G_+(x)=G_-(x)\ ,\quad G_{11}(xy)=0,\quad G_{++}(xy)+G_{--}(xy)=G_{+-}(xy)+G_{-+}(xy) \tag{2.38}$$

or

$$G_F+G_{\tilde{F}}=G_++G_-\ . \tag{2.39}$$

b. Theorem 2. The other components of $\tilde{G}$ can be expressed as

$$\tilde{G}_{2\cdots2(k),1\cdots1(n-k)}(12\cdots n)=(-i)^{n-1}\sum_{P\begin{pmatrix}12\cdots n\\ p_1p_2\cdots p_n\end{pmatrix}}{}'\ \Theta(p_1p_2\cdots p_n)\langle((\cdots(\phi(p_1),\phi(p_2)),\cdots),\phi(p_n))\rangle\ ,$$

where

$$(\ldots,\varphi(p_i))=\begin{cases}[\ ,\varphi(p_i)], & \text{if } k+1\le p_i\le n\\ \{\ ,\varphi(p_i)\}, & \text{if } 1\le p_i\le k\ ,\end{cases} \tag{2.40}$$

the prime over summation indicating that the cases $k+1\le p_i\le n$ are excluded from the possible permutations. The component $\tilde{G}_{2\cdots2}(1\cdots n)$ corresponds to the case $n=k$. All the other components of $\tilde{G}$ are defined as the results of the symmetry properties of the CTPGF's

$$G_{\cdots1\cdots2\cdots}(\cdots i\cdots j\cdots)=G_{\cdots2\cdots1\cdots}(\cdots j\cdots i\cdots)\ . \tag{2.41}$$

For the two-point functions we obtain

$$G_{21}(xy) \equiv G_r(xy) = -i\Theta(t_x,t_y)\langle[\varphi(x),\varphi(y)]\rangle \ ,$$
$$G_{12}(xy) \equiv G_a(xy) = -i\Theta(t_y,t_x)\langle[\varphi(y),\varphi(x)]\rangle \ ,$$
$$G_{22}(xy) \equiv G_c(xy) = -i\langle\{\varphi(x),\varphi(y)\}\rangle \ , \quad (2.42)$$

or in the matrix form

$$\tilde{G}(xy) = \begin{pmatrix} 0 & G_a \\ G_r & G_c \end{pmatrix} . \quad (2.43)$$

Making use of Eq. (2.39), we obtain

$$G_r = G_F - G_+ = G_- - G_{\tilde{F}} \ ,$$
$$G_a = G_F - G_- = G_+ - G_{\tilde{F}} \ , \quad (2.44)$$
$$G_c = G_F + G_{\tilde{F}} = G_+ + G_- \ .$$

The inverse transformation is given by Eq. (2.36) and can be written as

$$\hat{G}(xy) = \tfrac{1}{2}G_c(xy)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \tfrac{1}{2}G_r(xy)\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \tfrac{1}{2}G_a(xy)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} . \quad (2.45)$$

The first theorem is valid for both connected and disconnected CTPGF's while the second one in general form is applicable only to the disconnected functions. Further details will be given in Appendix B.

These relationships among different forms of the CTPGF's will be found quite useful in the applications. Here we shall illustrate them by several simple examples and derive some additional computational rules.

The simplest case for the connection of the CTPGF's "in sequence" is the integral of two single point functions

$$\int_p J_p(x)\varphi_p(x) = \int_{-\infty}^{\infty} \hat{J}\sigma_3\hat{\varphi} = 2\int_{-\infty}^{\infty} \tilde{J}\sigma_1\tilde{\varphi} \ , \quad (2.46)$$

where $\hat{J} = (J_+, J_-)$, $\tilde{\varphi}^T = (\varphi_1, \varphi_2)$, etc. For short, in what follows we shall omit the symbol of integration. The integral for G_p can be understood only in the coordinate presentation, while that for $\hat{G}$ and $\tilde{G}$ can be written in both coordinate and momentum space.

The linear response to the external source can be written as

$$R_p(1) = G_p(12)J_p(2)$$

or

$$\hat{R}(1) = \hat{G}(12)\sigma_3\hat{J}(2) \quad (2.47)$$

or

$$\tilde{R}(1) = \tilde{G}(12)\sigma_1\tilde{J}(2) \ .$$

If we take $J_+(x) = J_-(x) = J(x)$, then

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} 0 \\ G_r J \end{pmatrix} , \quad (2.48)$$

where the retarded function does appear naturally.

For the product of two-point functions we obtain

$$D_p(12) = A_p(13)B_p(32) \ ,$$
$$\hat{D}(12) = \hat{A}(13)\sigma_3\hat{B}(32) \ , \quad (2.49)$$
$$\tilde{D}(12) = \tilde{A}(13)\sigma_1\tilde{B}(32) \ ,$$

or in components

$$\begin{pmatrix} 0 & D_a \\ D_r & D_c \end{pmatrix} = \begin{pmatrix} 0 & A_a B_a \\ A_r B_r & A_r B_c + A_c B_a \end{pmatrix} . \quad (2.50)$$

This rule can be generalized to the multiple product with

$$Z_p = A_p^{(1)}A_p^{(2)} \cdots A_p^{(n)} \ ,$$
$$\hat{Z} = \hat{A}^{(1)}\sigma_3\hat{A}^{(2)} \cdots \sigma_3\hat{A}^{(n)} \ , \quad (2.51)$$
$$\tilde{Z} = \tilde{A}^{(1)}\sigma_1\tilde{A}^{(2)} \cdots \sigma_1\tilde{A}^{(n)} \ .$$

The last equation can be written in components as

$$Z_r = A_r^{(1)}A_r^{(2)} \cdots A_r^{(n)} \ ,$$
$$Z_a = A_a^{(1)}A_a^{(2)} \cdots A_a^{(n)} \ , \quad (2.52)$$
$$Z_c = \sum_{k=1}^{n} A_r^{(1)} \cdots A_r^{(k-1)}A_c^{(k)}A_a^{(k+1)} \cdots A_a^{(n)} \ .$$

Similarly, by use of the inverse transformation (2.36) and (2.45) we obtain

$$Z_\mu = \sum_{k=1}^{n} A_r^{(1)} \cdots A_r^{(k-1)}A_\mu^{(k)}A_a^{(k+1)} \cdots A_a^{(n)} \ , \quad (2.53)$$

where $\mu = +-$ or $-+$.

If the multipoint functions stand under the integration, attention has to be paid to the order of the variables. For example the three-point vertex function

$$\Gamma_p(123) = i\Gamma_p(14)\Gamma_p(25)\Gamma_p(36)G_p(456)$$

becomes

$$\hat{\Gamma}(123) = i(\hat{\Gamma}\sigma_3)(14)(\hat{\Gamma}\sigma_3)(25)(\hat{\Gamma}\sigma_3)(36)\hat{G}(456) \quad (2.54)$$

and then

$$\tilde{\Gamma}(123) = i(\tilde{\Gamma}\sigma_1)(14)(\tilde{\Gamma}\sigma_1)(25)(\tilde{\Gamma}\sigma_1)(36)\tilde{G}(456)$$

or in components

$$\Gamma_{111} = 0 \ ,$$
$$\Gamma_{211} = i\Gamma_r\Gamma_a\Gamma_a G_{211} \ ,$$
$$\Gamma_{221} = i(\Gamma_c\Gamma_r\Gamma_a G_{121} + \Gamma_r\Gamma_c\Gamma_a G_{211} + \Gamma_r\Gamma_r\Gamma_a G_{221})$$

No additional numerical coefficient appears after transformation in any inherent to the theory relations among the multipoint CTPGF's. For example, the four-point vertex function $\Gamma_p^{(4)}$ is related to the amputated Green's functions W_p as[9]

$$\Gamma_p^{(4)} = -W_p^{(4)} + 3W_p^{(3)}G_p^{(2)}W_p^{(3)} .$$

It can be transformed into

$$\hat{\Gamma}^{(4)} = -\hat{W}^{(4)} + 3\hat{W}^{(3)}\sigma_3\hat{G}^{(2)}\sigma_3\hat{W}^{(3)}$$

and

$$\tilde{\Gamma}^{(4)} = -\tilde{W}^{(4)} + 3\tilde{W}^{(3)}\sigma_1\tilde{G}^{(2)}\sigma_1\tilde{W}^{(3)} .$$

Contrary to this, a numerical factor may appear in some relations obtained by an artificial contraction. For example, the relation

$$A_p(12) = B_p(134)C_p(342)$$

becomes

$$\tilde{A}(12) = \tfrac{1}{2}\tilde{B}(134)(\sigma_1)_{33'}(\sigma_1)_{44'}\tilde{C}(3'4'2)$$

with the coefficient $\frac{1}{2}$. These examples justify the choice of the numerical constant $2^{n/2-1}$ in the transformation formula from $\hat{G}$ to $\tilde{G}$ [Eq. (2.35)].

The δ function on the closed time path $\delta_p(x-y)$ can be written in the matrix form as

$$\hat{\delta}(x-y) = \delta(x-y)\sigma_3 \tag{2.55}$$

after transformation it becomes

$$\tilde{\delta}(x-y) = Q\hat{\delta}(x-y)Q^{-1} = \delta(x-y)\sigma_1 . \tag{2.56}$$

They are the time derivatives for the step functions on the closed time path, which in the matrix form are

$$\hat{\Theta}(12) = \begin{pmatrix} \Theta(12) & 0 \\ 1 & \Theta(21) \end{pmatrix} \tag{2.57}$$

and

$$\tilde{\Theta}(12) = \begin{pmatrix} 0 & -\Theta(21) \\ \Theta(12) & 1 \end{pmatrix} . \tag{2.58}$$

The Dyson equation for the CTPGF's Eq. (2.22) can be rewritten as

$$\hat{G}\sigma_3\hat{\Gamma} = \hat{\Gamma}\sigma_3\hat{G} = -\hat{\delta} , \quad \tilde{G}\sigma_1\tilde{\Gamma} = \tilde{\Gamma}\sigma_1\tilde{G} = -\tilde{\delta} . \tag{2.59}$$

It is interesting to note that all characteristic features of $\hat{G}$ and $\tilde{G}$ are "transmitted" automatically to $\hat{\Gamma}$ and $\tilde{\Gamma}$. In particular, we have

$$\Gamma_{11} = 0, \quad \Gamma_{++} + \Gamma_{--} = \Gamma_{+-} + \Gamma_{-+} , \tag{2.60}$$

$$\tilde{\Gamma} = \begin{pmatrix} 0 & \Gamma_a \\ \Gamma_r & \Gamma_c \end{pmatrix} , \tag{2.61}$$

$$\Gamma_a = -G_a^{-1}, \quad \Gamma_r = -G_r^{-1}, \quad \Gamma_c = G_r^{-1}G_cG_a^{-1} . \tag{2.62}$$

It is easy to show from the symmetry properties that for the Hermitian Bose field

$$\hat{G} = (\hat{G})^T = -\sigma_1\hat{G}^*\sigma_1 = -\sigma_1\hat{G}^\dagger\sigma_1 , \tag{2.63a}$$

which after the Fourier transformation becomes

$$\hat{G}(k) = [\hat{G}(-k)]^T = -\sigma_1\hat{G}^*(-k)\sigma_1 = -\sigma_1\hat{G}^\dagger(k)\sigma_1 \tag{2.63b}$$

or

$$\tilde{G}(k) = \tilde{G}^T(-k) = -\sigma_3\tilde{G}^*(-k)\sigma_3 = -\sigma_3\tilde{G}^\dagger(k)\sigma_3 , \tag{2.63c}$$

where T means transposition, $*$ complex conjugation, and $\dagger$ Hermitian conjugation. All these properties are transmitted to $\hat{\Gamma}$ and $\tilde{\Gamma}$ through Eq. (2.59). Similarly, the specific features of multipoint Green's functions are conveyed to the corresponding vertex functions through relations like Eq. (2.54).

The transitivity of the CTPGF's also holds for some connection "in parallel," i.e., the product of several CTPGF's connecting two points, which itself is the constituent of a Green's function. Consider for example

$$S_p(12) \equiv G_p^3(12) . \tag{2.64}$$

It may be a self-energy part of G_p. In fact by use of Eq. (2.39) and $G_r(12)G_a(12)=0$ we obtain

$$G_{++}^3(12) + G_{--}^3(12) = G_{+-}^3(12) + G_{-+}^3(12) . \tag{2.65}$$

Moreover, the matrix

$$\tilde{S}(12) = \frac{1}{4}\begin{pmatrix} 0 & G_a(G_a^2+3G_c^2) \\ G_r(G_r^2+3G_c^2) & G_c[G_c^2+3(G_r+G_a)^2] \end{pmatrix} \tag{2.66}$$

behaves much like a simple G.

C. Further properties of two-point functions

By use of Eqs. (2.59)–(2.61) and (2.36) the two point vertex function $\hat{\Gamma}$ can be presented as

$$\hat{\Gamma} = -iB(I+\sigma_i) - A\sigma_2 - D\sigma_3 \tag{2.67}$$

with

$$B = \tfrac{1}{2}i(\Gamma_F + \Gamma_{\tilde{F}}) = \tfrac{1}{2}i(\Gamma_+ + \Gamma_-) , \tag{2.68a}$$

$$D = \tfrac{1}{2}(\Gamma_{\tilde{F}} - \Gamma_F) = -\tfrac{1}{2}(\Gamma_r + \Gamma_a) , \tag{2.68b}$$

$$A = \tfrac{1}{2}i(\Gamma_- - \Gamma_+) = \tfrac{1}{2}i(\Gamma_r - \Gamma_a) , \tag{2.68c}$$

where B, D, and A, are Hermitian matrices in the multicomponent fields or in the coordinate presentation. Equations (2.68b) and (2.68c) can be rewritten as

$$\Gamma_r = -G_r^{-1} = -D - iA \ , \quad (2.69a)$$

$$\Gamma_a = -G_a^{-1} = -D + iA \ . \quad (2.69b)$$

We shall call D the dispersion part, which determines the energy spectrum of the quasiparticle, and call A the dissipative part, which describes the decay rate of the elementary excitation.

The solution of the Dyson equation (2.59) can be presented as

$$\hat{G} = -\tfrac{1}{2}(\Gamma_r^{-1}\hat{N}_r - \hat{N}_a\Gamma_a^{-1}) = \tfrac{1}{2}(G_r\hat{N}_r - \hat{N}_a G_a) \ , \quad (2.70)$$

with

$$\hat{N}_r = (I+\sigma_1)(N+\sigma_3), \quad \hat{N}_a = (N-\sigma_3)(I+\sigma_1) \ , \quad (2.71)$$

where N is a matrix satisfying the equation

$$N\Gamma_a - \Gamma_r N = 2iB \ , \quad (2.72a)$$

or

$$ND - DN = i(NA + AN) - 2iB \ . \quad (2.72b)$$

The causal propagator of the quasiparticle with energy $\epsilon(\vec{k})$, corresponding to the pole of G_r and G_a, can be presented through the density operator n as

$$G_F = G_r(1+n) - nG_a \ . \quad (2.73)$$

By comparison with Eq. (2.70) we obtain

$$N|_{\text{pole}} = 1 + 2n \ . \quad (2.74)$$

In terms of n Eq. (2.72b) becomes

$$nD - Dn = i(nA + An) + i(A - B) = i(nA + An) + \Gamma_+ \quad (2.75)$$

which is the kinetic equation for the quasiparticle density n.[5] The right-hand side of Eq. (2.75) can be presented as

$$(1+n)\Gamma_+ - n\Gamma_- \ , \quad (2.76)$$

if the noncommutativity of n and A is ignored. It corresponds to the collision term in the kinetic equation which vanishes in the thermal equilibrium; i.e.,

$$\frac{\Gamma_-}{\Gamma_+} = \frac{1+n}{n} = e^{\beta\epsilon(\vec{k})} \ . \quad (2.77)$$

It can be shown[5] that $i\Gamma_\pm = i\Sigma_\pm$ where $\Sigma_\pm$ is the proper self-energy part, which itself is proportional to the probability of emission (+) or absorption (−) of the quasiparticles per unit time so

$$i\Gamma_\pm > 0 \ . \quad (2.78)$$

Equation (2.77) shows that in thermal equilibrium the probability of absorption is greater than that of emission as expected.

D. Path integral presentation and Ward-Takahashi identities

Suppose the Lagrangian of the system is globally invariant under the Lie group G which may contain the space-time symmetry group as its subgroup. Let $\varphi(x)$ be the basic fields, $Q(x)$ order parameters, which are functions of $\varphi(x)$. Both $\varphi(x)$ and $Q(x)$ have several components forming the bases of the unitary representation for G.

Under the infinitesimal transformations of G

$$\varphi(x) \to \varphi'(x) = \varphi(x) + \delta\varphi(x) \ ,$$
$$\delta\varphi(x) = \zeta_\alpha[i\hat{I}_\alpha^0 - X_\alpha^\mu(x)\partial_\mu]\varphi(x) = i\hat{I}_\alpha\varphi(x)\zeta_\alpha \ , \quad (2.79)$$

and

$$Q(x) \to Q'(x) = Q(x) + \delta Q(x) \ ,$$
$$\delta Q(x) = \zeta_\alpha[i\hat{L}_\alpha^0 - X_\alpha^\mu(x)\partial_\mu]Q(x) = i\hat{L}_\alpha Q(x)\zeta_\alpha \ , \quad (2.80)$$

where ζ_α are a total of n_G infinitesimal parameters for group G and $\hat{I}_\alpha^0$, $\hat{L}_\alpha^0$ are Hermitian representation matrices for the generators of G. $X_\alpha^\mu(x)$ are associated with the transformations of coordinates

$$X^\mu \to X^{\mu\prime} = X^\mu + X_\alpha^\mu(x)\zeta_\alpha \ . \quad (2.81)$$

It can be shown easily that the Lagrangian function transforms as

$$\mathcal{L}(\varphi'(x))\frac{d^4x}{d^4x'} = \mathcal{L}(\varphi(x')) + \left[\frac{\delta\mathcal{L}}{\delta\varphi(x)} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)}\right]\delta\varphi(x) + \partial_\mu[j_\alpha^\mu(x)\zeta_\alpha(x)] \ , \quad (2.82)$$

where

$$j_\alpha^\mu(x) = i\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)}\hat{I}\varphi(x) + \mathcal{L}X_\alpha^\mu(x) \quad (2.83)$$

is the current in direction α. If the Lagrangian is invariant under the global transformation of G it follows that

$$\partial_\mu j_\alpha^\mu(x) = i\left[\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi(x)} - \frac{\delta\mathcal{L}}{\delta\varphi(x)}\right]\hat{I}_\alpha\varphi(x) \ . \quad (2.84)$$

The Eq. (2.84) shows that the currents $j_\alpha^\mu(x)$ are conserved if $\varphi(x)$ is the solution of the Euler-Lagrangian equation. By use of Eq. (2.84), Eq. (2.82) can be rewritten as

$$\mathcal{L}(\varphi'(x))\frac{d^4x}{d^4x'} = \mathcal{L}(\varphi(x')) + j_\alpha^\mu(x)\partial_\mu\zeta_\alpha(x) \ . \quad (2.85)$$

This is the transformation of the Lagrangian under the local action of G, if it is invariant under the global action of the same group.

The generating functional for the CTPGF's can be presented in the form of a Feynman path integral by the well-known procedure in the field theory

$$Z(h(x),J(x)) = N\int [d\varphi(x)]\exp\left[i\int_p [\mathcal{L}(\varphi(x)) - h(x)\varphi(x) - J(x)Q(x)]\right]\langle\varphi(\vec{x},t_+=-\infty|\hat{\rho}|\varphi(\vec{x},t_-=-\infty)\rangle \tag{2.86}$$

N being the normalization constant. What is different from the path integral in the ordinary field theory is that the integration is carried out over the closed time path and that the boundary conditions are determined by the density matrix $\hat{\rho}$.

Transforming integration variable in Eq. (2.86) from $\varphi(x)$ to $\varphi'(x)$ under the local action of group G with infinitesimal parameters $\zeta_\alpha(x)$, satisfying boundary conditions

$$\zeta_\alpha(\vec{x},t_\pm=-\infty)=0\ ,\quad \lim_{|\vec{x}|\to\infty}\zeta_\alpha(\vec{x},t)=0\ , \tag{2.87}$$

taking into account that the measure $[d\varphi(x)]$ does not change under the unitary transformation and that the matrix element of $\hat{\rho}$ remains the same as a result of Eq. (2.87), we obtain

$$\partial_\mu j^\mu_\alpha\left[\varphi(x)=i\frac{\delta}{\delta h(x)}\right]Z(h(x),J(x)) = \left[h(x)\hat{I}_\alpha\frac{\delta}{\delta h(x)} + J(x)\hat{L}_\alpha\frac{\delta}{\delta J(x)}\right]Z(h(x),J(x))\ . \tag{2.88}$$

By use of the commutation relation

$$i\frac{\delta}{\delta h(x)}Z = Z\left[\varphi_c(x)=i\frac{\delta}{\delta h(x)}\right] \tag{2.89}$$

Eq. (2.88) can be rewritten as

$$\begin{aligned}\partial_\mu\langle j^\mu_\alpha(x)\rangle &= \partial_\mu j^\mu_\alpha[\varphi_c(x)+i\delta/\delta h(x)]\\ &= -i[h(x)\hat{I}_\alpha\varphi_c(x)+J(x)\hat{L}_\alpha Q_c(x)]\ .\end{aligned} \tag{2.90}$$

This is the required Ward-Takahashi identity which has the same form as in the usual field theory, but here x can take arbitrary value on the closed time path.

Introducing the generating functional for the connected CTPGF's

$$W(h(x),J(x)) = i\ln Z(h(x),J(x))\ , \tag{2.91}$$

and the vertex function

$$\begin{aligned}\Gamma(\varphi_c(x),Q_c(x)) &= W(h(x),J(x))\\ &\quad -\int_p [h(x)\varphi_c(x)+J(x)Q_c(x)]\ ,\end{aligned} \tag{2.92}$$

where

$$\varphi_c(x)=\delta W/\delta h(x),\quad Q_c(x)=\delta W/\delta J(x)\ , \tag{2.93}$$

we obtain from Eq. (2.90)

$$\begin{aligned}&\partial_\mu j^\mu_\alpha\left[\frac{\delta W}{\delta h(x)}+i\frac{\delta}{\delta h(x)}\right]\\ &\quad = -i\left[h(x)\hat{I}_\alpha\frac{\delta W}{\delta h(x)}+J(x)\hat{L}_\alpha\frac{\delta W}{\delta J(x)}\right]\end{aligned} \tag{2.94}$$

and

$$\begin{aligned}&\partial_\mu j^\mu_\alpha\left[\varphi_c(x)+i\int_p\left[\frac{\delta h(x)}{\delta\varphi_c(y)}\right]^{-1}\frac{\delta}{\delta\varphi_c(y)}\right]\\ &\quad = i\left[\frac{\delta\Gamma}{\delta\varphi_c(x)}\hat{I}_\alpha\varphi_c(x)+\frac{\delta\Gamma}{\delta Q_c(x)}\hat{L}_\alpha Q_c(x)\right]\ .\end{aligned} \tag{2.95}$$

Taking derivatives with respect to $h(x)$, $J(x)$ in Eq. (2.94) and then putting them to zero, we obtain successive WT identities for all orders of CTPGF's. The similar procedure in Eq. (2.95) with respect to $\varphi_c(x)$, $Q_c(x)$ will yield WT identities for the vertex functions.

The equations for the vertex functional Γ in the vanishing external field

$$\delta\Gamma/\delta\varphi_c(x)=0,\quad \delta\Gamma/\delta Q_c(x)=0 \tag{2.96}$$

can be used to discuss the spontaneous symmetry breaking and the Goldstone mode.[5] It is worthwhile to note that with the fluctuation effects being taken into account, the Eq. (2.96) does not have stable solitonlike solution

$$Q_c(x)=Q_0(\vec{x})e^{-i\omega t}\ , \tag{2.97}$$

where $Q_0(\vec{x})$ is different from zero in a limited domain of space, or the laser type solution with

$$Q_0(\vec{x})=e^{i\vec{k}\cdot\vec{x}}\ . \tag{2.98}$$

Up to now we have considered only the linear transformations of fields under the action of symmetry group. In critical dynamics the nonlinear transformations are also needed.

Suppose $\varphi_i(x)$ are basic fields, transforming under the action of an internal symmetry group G (i.e., the space-time coordinates are not involved) like

$$\varphi_i(x) \to \varphi_i' = \varphi_i + A_{i\alpha}(\varphi)\zeta_\alpha \ , \tag{2.99}$$

where ζ_α remain infinitesimal group parameters, but contrary to the previous case, here $A_{i\alpha}(\varphi)$ may be arbitrary function of φ.

If the Lagrangian is invariant under the global transformation of G, we have

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + j_\alpha^\mu \partial_\mu \zeta_\alpha(x) \ , \tag{2.100}$$

where

$$j_\alpha^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi_i} A_{\alpha i}(\varphi) \ . \tag{2.101}$$

The Ward-Takahashi identity in this case can be derived also from the path integral presentation of the generating functional, but an additional term comes from the Jacobian of transformation. We have

$$\begin{aligned}\int [d\varphi_i] \exp\left[i\int_p (\mathcal{L} - J\varphi)\right]\langle|\rho|\rangle &= \int [d\varphi_i'] \exp\left[i\int_p [\mathcal{L} + j_\alpha^\mu \partial_\mu \zeta_\alpha - J_i A_{\alpha i}(\varphi)\zeta_\alpha - J\varphi]\right]\langle|\rho|\rangle \\ &= \int [d\varphi_i]\left[1 + \frac{\partial A_{\alpha i}(\varphi)}{\partial \varphi_i}\zeta_\alpha\right]\left[1 + \int_p [j_\alpha^\mu \partial_\mu \zeta_\alpha - J_i A_{\alpha i}(\varphi)\zeta_\alpha]\right] \\ &\quad \times \exp\left[i\int (\mathcal{L} - J\varphi)\right]\langle|\rho|\rangle \ ,\end{aligned} \tag{2.102}$$

where

$$\langle|\rho|\rangle \equiv \langle \varphi(\vec{x}, t_+ = -\infty)|\rho|\varphi(\vec{x}, t_- = -\infty)\rangle \ .$$

It follows from Eq. (2.102) that

$$\partial_\mu j_\alpha^\mu \left[\varphi_{ic}(x) + i\frac{\delta}{\delta J_i(x)}\right] = \frac{\partial A_{\alpha i}}{\partial \varphi_i}\left[\varphi_{jc} + i\frac{\delta}{\delta J_j(x)}\right] - J_i A_{\alpha i}\left[\varphi_{jc} + i\frac{\delta}{\delta J_j(x)}\right] \ . \tag{2.103}$$

If the loop correction terms are neglected, Eq. (2.103) turns out to be

$$\partial_\mu \langle j_\alpha^\mu(x)\rangle = \partial A_{\alpha i}/\partial \varphi_i - J_i A_{\alpha i} \ . \tag{2.104}$$

This equation will be used to obtain the nonlinear mode coupling term in the generalized Langevin equation.

III. SUMMARY OF CRITICAL DYNAMICS

There was a recent comprehensive review on the critical dynamics.[10] We give here a brief summary of the basic results to specify the notations and to facilitate the comparison with our results.

The properties of the system near critical point are described in terms of order parameters and conserved variables forming a set of macrovariables $Q = \{Q_i, i = 1, 2 \cdots n\}$. The time evolution of these stochastic variables obeys the generalized Langevin equation

$$\partial Q_i(t)/\partial t = K_i(Q) + \xi_i(t) \ , \tag{3.1}$$

where the random force $\xi_i(t)$, reflecting the effects of all degrees of freedom, not included in $\{Q_i\}$, is assumed to be Gaussian distributed, i.e.,

$$\langle \xi_i(t)\rangle = 0 \ , \ \langle \xi_i(t)\xi_j(t')\rangle = 2\sigma_{ij}\delta(t-t') \ . \tag{3.2}$$

The right-hand side function $K_i(Q)$ of Eq. (3.1) consists of two parts

$$K_i(Q) = -\sigma_{ij}\delta F/\delta Q_j + V_i(Q) \ , \tag{3.3}$$

where the free energy $F \equiv F(Q)$ as a functional of Q is dependent on concrete models. The static equilibrium condition $\delta F/\delta Q_i = 0$ appears to be the Ginzburg-Landau equation. Therefore Eq. (3.1) without random force ξ_i is called sometimes the time-dependent Ginzburg-Landau equation (TDGL for short). In principle, the coefficient matrix σ_{ij} may have both symmetric and antisymmetric parts. The symmetric part describes the relaxation, while the antisymmetric one describes the canonical motion. If only relaxation effects are considered, σ_{ij} may be taken symmetric. According to the fluctuation dissipation theorem, the same matrix σ_{ij} does appear both in Eqs. (3.2) and (3.3). In diagonalized form $\sigma_i = $constant (dissipative relaxation) for the nonconserved Q_i, and $\sigma_i = -D_i\nabla^2$ (diffusion relaxation, D_i being the diffusion constant) for the conserved Q_i.

The dissipative coupling of different modes can be described by means of the interaction terms in the free-energy functional, but the reversible mode coupling appears as stream term $V_i(Q)$ in Eq. (3.3). Usually it takes the form[11]

$$V_i(Q) = \lambda \sum_j \left[\frac{\partial}{\partial Q_j} A_{ij}(Q) - A_{ij}(Q)\frac{\delta F}{\delta Q_j}\right] \ , \tag{3.4}$$

where the antisymmetric tensor A_{ij} is formed from the commutators or the Poisson brackets. As a rule, the linear approximation is accepted, i.e.,

$$A_{ij}=f_{ijk}Q_k \ , \tag{3.5}$$

where f_{ijk} are structure constants for the underlying symmetry group. The form of the expression (3.4) itself makes the conservation equation for probability to be satisfied, i.e.,

$$(\partial/\partial Q_i)[V_i(Q)e^{-F}]=0 \ , \tag{3.6}$$

which means that the $\{Q_i\}$ space is divergence free, ensuring e^{-F} to be the stationary distribution with detailed balance.

It can be seen that the Langevin equation (3.1) is flexible enough to embody all possible factors. To our knowledge Eq. (3.1) is "assembled" by different reasonable arguments, so that it remains a kind of phenomenological model.

The widely accepted approach in the theory of dynamical critical phenomena is to construct the perturbation theory by iterating Eq. (3.1).[12] Since there are two different kinds of "constituent parts"—response and correlation functions, the structure of the perturbation theory becomes quite complicated. The compact presentation of such perturbation procedure is given by the MSR field theory[8] mentioned before. In analogy with the static theory of original K. G. Wilson's formulation, Eq. (3.1) can be used to carry out the renormalization transformation to derive the recurrent formulas and to calculate the critical exponents.

For the last several years the critical dynamics has been reformulated using the field-theoretical approach.[13,14]

The Gaussian stochastic process $\xi_i(t)$ can be presented by a stochastic functional.[15] Equation (3.1) can be considered as a mapping of the Gaussian process $\xi_i(t)$ onto a more complicated process $Q_i(t)$. Performing such nonlinear transformation of the Gaussian stochastic functional yields the functional description for process $Q_i(t)$.[16] A more direct way is to start from the normalization condition for δ functions under the path integral

$$\int [dQ]\delta\left[\frac{\partial Q}{\partial t}-K(Q)-\xi\right]\Delta(Q)=1 \ . \tag{3.7}$$

Since the argument of δ function is not Q, but the whole expression (3.1) it is necessary to insert the Jacobian $\Delta(Q)$ for the nonlinear transformation from ξ_i to Q_i. Neglecting multiplicative constant $\Delta(Q)$ turns out to be[16]

$$\Delta(Q)=\exp\left[-\tfrac{1}{2}\int\frac{\delta K(Q)}{\delta Q}dx\right] \ , \tag{3.8}$$

where $dx=d\vec{x}dt$ is the four-dimensional integration element. In what follows we shall omit dx for short.

Presenting the δ functions in Eq. (3.7) in terms of the continuous integral, we obtain

$$\int [dQ]\left[\frac{d\hat{Q}}{2\pi}\right]\exp\left\{\int\left[i\hat{Q}\left[\frac{\partial Q}{\partial t}-K(Q)-\xi\right]-\frac{1}{2}\frac{\delta K}{\delta Q}\right]\right\}=1 \ . \tag{3.9}$$

The insertion of factor

$$\exp\left\{-i\int[J(x)Q(x)+\hat{J}(x)\hat{Q}(x)]\right\}$$

into the integral (3.9) yields the generating functional for the averages of all possible products of the field operators (in theory of probability it is called characteristic or moment-generating functional):

$$Z_\xi(J,\hat{J})=\int[dQ]\left[\frac{d\hat{Q}}{2\pi}\right]\exp\left\{\int\left[i\hat{Q}\left[\frac{\partial Q}{\partial t}-K(Q)-\xi\right]-\frac{1}{2}\frac{\delta K}{\delta Q}-i(JQ+\hat{J}\hat{Q})\right]\right\} \ , \tag{3.10}$$

with the obvious normalization condition

$$Z_\xi[0,0]=1 \ . \tag{3.11}$$

The random force $\xi(t)$ obeys the Gaussian distribution

$$W(\xi)\propto\exp(-\tfrac{1}{2}\xi\sigma^{-1}\xi) \ , \tag{3.12}$$

where σ^{-1} is the inverse of the correlation matrix σ. Taking average in Eq. (3.10) over ξ, we obtain the Lagrangian formulation of the generating functional for the classical statistical field theory

$$Z(J,\hat{J})=\int[dQ]\left[\frac{d\hat{Q}}{2\pi}\right]\exp\left\{\int\left[-\frac{1}{2}\hat{Q}\sigma\hat{Q}+i\hat{Q}\left[\frac{\partial Q}{\partial t}-K(Q)\right]-\frac{1}{2}\frac{\delta K}{\delta Q}-iJQ-i\hat{J}\hat{Q}\right]\right\} \ . \tag{3.13}$$

In their original paper MSR introduced the "response fields," $\hat{Q}$ in our notation, noncommutative with the basic fields, to simplify the structure of the perturbation theory and the renormalization procedure. As in the ordinary field theory, the noncommutativity of the variables is not evident under the path integration. The introduction of the $\hat{Q}$ fields doubles the number of operators. In the CTPGF approach the time path is divided into positive and negative branches, so the number of operators is also doubled. As will be shown in Sec. V, the Lagrangian formulation of the MSR field theory follows naturally from the CTPGF formalism. In can be seen also that the noncommutativity of operators is not an artificial formal trick, but a necessity to describe properly the statistical fluctuations.

The Gaussian integration over $\hat{Q}$ in Eq. (3.13) can be carried out to yield

$$Z[J]=\int[dQ]\exp\left\{\int\left[-\frac{1}{2}\left(\frac{\partial Q}{\partial t}-K(Q)-\hat{J}\right)\sigma^{-1}\left(\frac{\partial Q}{\partial t}-K(Q)-\hat{J}\right)-\frac{1}{2}\frac{\delta K}{\delta Q}-iJQ\right]\right\} . \quad (3.14)$$

This expression was first obtained as a stochastic functional in Ref. 16, but it turns out that Eq. (3.13) is the more convenient starting point for the critical dynamics.[13,14]

IV. FUNDAMENTAL SYSTEM OF EQUATIONS IN CRITICAL DYNAMICS

Generally speaking, both order parameter and conserved variables when regarded as macrovariables are composite operators of the basic fields. We shall specify them by somewhat different notation. The nonconserved order parameters may be written as

$$Q_{ci}(x),\quad i=1,2,\ldots,n ,$$

whereas the conserved variables

$$Q_{c,n+\alpha}(x)\equiv q_\alpha(x),\quad \alpha=1,2,\ldots,m ,$$

where q_α corresponds to the average of the zeroth component for the conserved current

$$q_\alpha=\langle j_\alpha^0\rangle . \quad (4.1)$$

Without sacrifice of generality both Q_{ci} and q_α can be taken to be Hermitian.

Introducing the generating functional of the CTPGF vertex functions for the composite operators $\Gamma(Q_c)$, we obtain the equations which require to be satisfied by Q_c [Eq. (2.21)]; i.e.,

$$\delta\Gamma/\delta Q_{ci}(x)=-J_i(x),\quad i=1,2,\ldots,n+m . \quad (4.2)$$

After taking the variational derivatives one puts $J_i(x+)=J_i(x-)=J_i(\vec{x},t)$, from which it follows [from Eq. (2.19)] that $Q_{ci}(x+)=Q_{ci}(x-)=Q_i(\vec{x},t)$, where $J_i(\vec{x},t)$, $Q_i(\vec{x},t)$ are functions defined on the usual time axis $(-\infty,+\infty)$. We next show that Eqs. (4.2) lead to the generalized TDGL equations under the assumption that Q_i are smoothly varying functions of time.

Suppose the macrovariables $Q_i(\vec{x},\tau)$ to be known at the moment τ. At the moment t following closely after τ the left-hand side of Eqs. (4.2) can be expanded. If x sits on the positive time branch, we have

$$-J_i(\vec{x},t)=\frac{\delta\Gamma}{\delta Q_{ci}(x+)}\Bigg|_{Q_{ci}(x+)=Q_{ci}(x-)=Q_i(\vec{x},\tau)}+\int\Gamma_{rij}(x,y)[Q_j(\vec{y},t_y)-Q_j(\vec{y},\tau)]dy , \quad (4.3)$$

where $\Gamma_{rij}(x,y)$ are two-point retarded vertex functions after taking $Q_{c+}=Q_{c-}=Q$. If x sits on the negative branch of time, the same is true due to Eq. (2.60) $\Gamma_r=\Gamma_{++}-\Gamma_{+-}=\Gamma_{-+}-\Gamma_{--}$. Since $Q_j(\vec{y},t)$ in Eq. (4.3) varies smoothly with time, to the first order of $(t_y-\tau)$ we have

$$Q_j(\vec{y},t_y)-Q_j(\vec{y},\tau)=(t_y-\tau)\partial Q_j(\vec{y},\tau)/\partial\tau . \quad (4.4)$$

Substituting Eq. (4.4) back into Eq. (4.3) and taking into account that in the limit $t\equiv t_x\rightarrow\tau$

$$\gamma_{ij}(\vec{x},\vec{y},\tau)\equiv-\lim_{t_x\rightarrow\tau}\int dt_y(t_y-t_x)\Gamma_{rij}(\vec{x},t_x,\vec{y},t_y)=i\frac{\partial}{\partial k_0}\Gamma_{rij}(\vec{x},\vec{y},k_0,\tau)|_{k_0=0} , \quad (4.5)$$

where $\Gamma_{rij}(\vec{x},\vec{y},k_0,\tau)$ are Fourier transforms with respect to (t_x-t_y) taken at the average time $T=\frac{1}{2}(t_x+t_y)\approx\tau$, we obtain

$$\gamma(\tau)\frac{\partial Q(\tau)}{\partial\tau}=\frac{\delta\Gamma}{\delta Q_{c+}}\bigg|_{Q_{c+}=Q_{c-}=Q}+J(t) \quad . \tag{4.6}$$

Here the matrix notation is used and $\gamma_{ij}(\vec{x},\vec{y},\tau)$ are considered to be matrix elements with subscripts $i\,\vec{x}$ and $j\,\vec{y}$.

For the moment let

$$I_i(\vec{x},\tau)\equiv\frac{\delta\Gamma}{\delta Q_{c+}}\bigg|_{Q_{c+}=Q_{c-}=Q(\tau)} \tag{4.7}$$

and we calculate the functional derivative of I_i, considering it as a functional of functions $Q(\vec{x},\tau)$ with three-dimensional argument $\vec{x}$

$$\frac{\delta I_i(\vec{x},\tau)}{\delta Q_j(\vec{y},\tau)}=\int d\vec{z}\,d\tau_z\left\{\frac{\delta^2\Gamma}{\delta Q_{ci}(x+)\delta Q_{ck}(z+)}\frac{\delta Q_{ck}(z+)}{\delta Q_j(\vec{y},\tau)}-\frac{\delta^2\Gamma}{\delta Q_{ci}(x+)\delta Q_{ck}(z-)}\frac{\delta Q_{ck}(z-)}{\delta Q_j(\vec{y},\tau)}\right\}\bigg|_{Q_{c+}=Q_{c-}=Q(\tau)} ,$$

where

$$\frac{\delta Q_{ck}(z)}{\delta Q_j(\vec{y},\tau)}\bigg|_{Q_{c+}=Q_{c-}=Q(\tau)}=\delta_{jk}\delta^{(3)}(\vec{y}-\vec{z}) \quad .$$

Thus we obtain

$$\delta I_i(\vec{x},\tau)/\delta Q_j(\vec{y},\tau)=\Gamma_{++ij}(\vec{x},\vec{y},k_0=0,\tau)-\Gamma_{+-ij}(\vec{x},\vec{y},k_0=0,\tau) \quad ,$$

where the $k_0=0$ components of Fourier transforms appear as in Eq. (4.5). It can be shown in the same way that

$$\delta I_j(\vec{y},\tau)/\delta Q_i(\vec{x},\tau)=\Gamma_{++ji}(\vec{y},\vec{x},k_0=0,\tau)-\Gamma_{+-ji}(\vec{y},\vec{x},k_0=0,\tau)=\Gamma_{++ij}(\vec{x},\vec{y},-k_0=0,\tau)-\Gamma_{-+ij}(\vec{x},\vec{y},-k_0=0,\tau) \quad ,$$

where the symmetry properties of Γ following from Eqs. (2.59) and (2.63) are used. The difference

$$\frac{\delta I_i(\vec{x},\tau)}{\delta Q_j(\vec{y},\tau)}-\frac{\delta I_j(\vec{y},\tau)}{\delta Q_i(\vec{x},\tau)}=\lim_{k_0\to 0}[\Gamma_{-+ij}(\vec{x},\vec{y},-k_0,\tau)-\Gamma_{+-ij}(\vec{x},\vec{y},k_0,\tau)] \tag{4.8}$$

vanishes due to Eq. (2.77), i.e.,

$$\Gamma_{+-}=\exp(-\beta k_0)\Gamma_{-+}$$

near thermal equilibrium, so that there exists a functional $F(Q_i(\vec{x},\tau))$ with

$$I_i(\vec{x},\tau)=-\delta F/\delta Q_i(\vec{x},\tau) \quad . \tag{4.9}$$

Equation (4.6) can be rewritten as

$$\gamma(\tau)\partial Q(\tau)/\partial\tau=-\delta F/\delta Q(\tau)+J(\tau) \quad . \tag{4.10}$$

If the macrovariables $Q(\tau)$ do not change with time in the external field J, i.e, in the stationary state, then

$$\delta F/\delta Q=J \quad . \tag{4.11}$$

Hence F is the effective free energy of the system and Eq. (4.11) is the Ginzburg-Landau equation, determining the stationary distribution of macrovariables.

For systems in stationary states far from equilibrium expression (4.8) is equal to zero only if the dissipative part of the vertex function $A=\frac{1}{2}i(\Gamma_{-+}-\Gamma_{+-})$ Eq. (2.68c) satisfies the condition

$$\lim_{k_0\to 0}A_{ij}(\vec{x},\vec{y},k_0,t)=0 \quad . \tag{4.12}$$

In this case I_i can also be written as a variational derivative of the free energy or effective potential. Some of the stationary states satisfying the so-called "potential conditions," provided by the detailed balance, as discussed by Graham and Haken,[17] must belong to this category.

In the vicinity of all stationary states with the potential functions F Eq. (4.10) constitutes the system of time-dependent GL equations, but they are much more general than the TDGL equations in the usual sense since the mode coupling terms are also included.

It is usually customary to multiply Eq. (4.10) by the inverse matrix $\gamma^{-1}(\tau)$ to obtain

$$\frac{\partial Q(\tau)}{\partial\tau}=\gamma^{-1}(\tau)\left[-\frac{\delta F}{\delta Q(\tau)}+J(\tau)\right] \quad . \tag{4.13}$$

Using the symmetry properties of the vertex functions, following from Eqs. (2.59) and (2.63)

$$\tilde{\Gamma}(k)=\tilde{\Gamma}^T(-k)=-\sigma_3\tilde{\Gamma}^*(-k)\sigma_3=-\sigma_3\tilde{\Gamma}^\dagger(k)\sigma_3 \ , \tag{4.14}$$

it can be shown that the real part of Γ_r is an even function of k_0, while the imaginary part is an odd one, so $\gamma(t)$ is a real matrix according to the definition (4.5).

In accordance with the numeration of the subscripts given at the beginning of this section, the matrix $\gamma(\tau)$ can be divided into four blocks. Two of them, corresponding to the conserved variables $\gamma_{\alpha\vec{x},\beta\vec{y}}$ and $\gamma_{\alpha\vec{x},j\vec{y}}$, can be fixed completely by comparison with the WT identities. For the general case, the proper form of the two blocks associated with the order parameters can be determined only by the symmetry considerations. This will be discussed below.

It is worthwhile to point out that Eq. (4.4) is equivalent to the Markovian approximation. In principle, the original Eq. (4.2) contains in itself the possibility of considering the memory effects.

Under the action of the symmetry group G of the system the conserved variables transform as the generators I^α of the group, i.e.,

$$q_\alpha \to q'_\alpha = q_\alpha + if_{\alpha\beta\gamma}\zeta_\beta q_\gamma \ , \tag{4.15}$$

where $f_{\alpha\beta\gamma}$ are the structure constants of the group and ζ_β are the infinitesimal parameters of transformation. The order parameters Q_i transform as some representation $\hat{L}$ of group G

$$Q_i \to Q_i' = Q_i + iL_{ij}^\alpha \zeta_\alpha Q_j \ . \tag{4.16}$$

As shown in Sec. II D, if the Lagrangian of the system is invariant under the global symmetry transformations, the WT identities (2.95) are valid on the closed time path. In the present case, Eq. (2.95) can be written as

$$\langle \partial_\mu j_\alpha^\mu(\varphi)\rangle = i\left[\frac{\delta\Gamma}{\delta Q_{ci}(x)}L_{ij}^\alpha Q_{cj}(x)+f_{\alpha\beta\gamma}\frac{\delta\Gamma}{\delta q_\beta(x)}q_\gamma(x)\right] \ , \tag{4.17}$$

where, as before, $\Gamma\equiv\Gamma(Q_{ci},q_\alpha)$ is the generating functional of the vertex CTPGF's for the composite operators. Putting $Q_{c+}=Q_{c-}=Q(\tau)$ and let $j_\alpha^\mu \equiv \langle j_\alpha^\mu(\varphi)\rangle$ in Eq. (4.17), we obtain

$$\partial q_\alpha/\partial\tau = \nabla\vec{j}_\alpha - i[J_i(\vec{x},\tau)L_{ij}^\alpha Q_j(\vec{x},\tau) + f_{\alpha\beta\gamma}J_\beta(\vec{x},\tau)q_\gamma(\vec{x},\tau)] \ , \tag{4.18}$$

where J, Q, q, etc., are functions defined on the usual time axis $(-\infty,+\infty)$.

To determine the conserved currents $\vec{j}_\alpha$ we perform the following manipulations in analogy with the procedure used to deal with the thermal perturbations in the linear-response theory. By introducing an additional artificial external source ΔJ, superposed on the original source J, the system is forced to come into the stationary state $\partial q_\alpha/\partial t=0$. In this case, $\vec{j}_\alpha$ in Eq. (4.18) changes to $\vec{j}'_\alpha$, so that

$$\nabla\vec{j}'_\alpha - i[(J_i+\Delta J_i)L_{ij}^\alpha Q_j + f_{\alpha\beta\gamma}(J_\beta+\Delta J_\beta)q_\gamma]=0 \ . \tag{4.19}$$

Since the system is in the stationary state, we can use Eq. (4.11), i.e.,

$$\delta F/\delta Q_i = J_i+\Delta J_i \ , \quad \delta F/\delta q_\alpha = J_\alpha + \Delta J_\alpha \ , \tag{4.20}$$

to replace the source terms J_i, J_α by the functional derivatives of the free energy F. According to the linear-response theory the difference between $\vec{j}'_\alpha$ and the conserved current $\vec{j}_\alpha$ without an artificial source ΔJ can be written as

$$\vec{j}'_\alpha = \vec{j}_\alpha - l_{\alpha\beta}\nabla(\delta F/\delta q_\beta - J_\beta) \ , \tag{4.21}$$

where $l_{\alpha\beta}$ are the linear transport coefficients. Substituting Eq. (4.21) into Eq. (4.19) yields

$$\nabla\vec{j}_\alpha = l_{\alpha\beta}\nabla^2\left[\frac{\delta F}{\delta q_\beta}-J_\beta\right] + i\left[\frac{\delta F}{\delta Q_i}L_{ij}^\alpha Q_j + f_{\alpha\beta\gamma}\frac{\delta F}{\delta q_\beta}q_\gamma\right] \ . \tag{4.22}$$

Inserting the expression (4.22) for $\nabla\vec{j}_\alpha$ back into Eq. (4.18) leads to the equation of motion of the conserved variables

$$\frac{\partial q_\alpha}{\partial\tau} = l_{\alpha\beta}\nabla^2\left[\frac{\delta F}{\delta q_\beta}-J_\beta\right] + i\left[\left[\frac{\delta F}{\delta Q_i}-J_i\right]L_{ij}^\alpha Q_j + f_{\alpha\beta\gamma}\left[\frac{\delta F}{\delta q_\beta}-J_\beta\right]q_\gamma\right] \ . \tag{4.23}$$

Comparing Eq. (4.23) with the general equations for macrovariables (4.13) in the case of conserved variables, one determines two blocks of γ^{-1} matrix

$$[\gamma^{-1}(\tau)]_{\alpha\vec{x},\beta\vec{y}} = -[l_{\alpha\beta}\nabla_x^2 + if_{\alpha\beta\gamma}q_\gamma(\vec{x},\tau)]\delta^{(3)}(\vec{x}-\vec{y}) \ , \tag{4.24}$$

$$[\gamma^{-1}(\tau)]_{\alpha\vec{x},i\vec{y}} = -iL_{ij}^\alpha Q_j(\vec{x},\tau)\delta^{(3)}(\vec{x}-\vec{y}) \ . \tag{4.25}$$

Now we turn to the equations for the order parameters. If the order parameters Q_i form the irreducible representation of the group G and take small values near the critical point, γ^{-1} can be expanded into a power series of Q_i. It follows from the symmetry property that

$$[\gamma^{-1}(\tau)]_{i\vec{x},j\vec{y}} = \delta_{ij}\sigma_{\vec{x},\vec{y}} + \cdots \ , \tag{4.26}$$

where $\sigma_{\vec{x},\vec{y}}$, not depending on Q_i, are determined by the kinetic and dissipative characteristics of the system. Similarly, by symmetry consideration another expansion can be written

$$[\gamma^{-1}(\tau)]_{i\vec{x},\alpha\vec{y}} = ifL_{ij}^\alpha Q_j(\vec{x},\tau)\delta^{(3)}(\vec{x}-\vec{y}) + \cdots \ , \tag{4.27}$$

where the ellipsis represents higher terms of Q_i and f is invariant under group transformations and can only be a numerical constant in the lowest order of Q_i. To determine the value of f we consider the limit of vanishing dissipation. In this case the antisymmetric part of the matrix $\gamma(\tau)$ in Eq. (4.10) is dominant. The same is true for the inverse matrix in Eq. (4.13). Comparison of Eqs. (4.27) and (4.25) gives

$$f=1 \ . \tag{4.28}$$

Since the first term of Eq. (4.25) is independent of dissipation, Eq. (4.28) remains valid in its presence. Generally speaking the expansions (4.26) and (4.27) may contain other terms, including the crossover interaction of the dissipative and canonical motion. We shall not touch this problem here. In the approximation discussed above we obtain the following equations for the order parameters:

$$\frac{\partial Q_i}{\partial \tau}=\sigma\left[\frac{\delta F}{\delta Q_i}-J_i\right]-iL_{ij}^{\alpha}Q_j\left[\frac{\delta F}{\delta q_\alpha}-J_\alpha\right] \ . \tag{4.29}$$

Equations (4.29) and (4.23) form the fundamental system of equations for the critical dynamics. In the CTPGF approach J, coming from $J_+=J_-$, is the real physical external field. It may contain the additional random fields, representing the effects of degrees of freedom, not included in the macrovariables Q_i. Therefore we call this system of equations the generalized Langevin equations.

The essential point of the above given derivation is that the mode coupling terms naturally appear in the generalized Langevin equations. Moreover, they actually have the form of Eqs. (3.4) and (3.5). To be more exact, the representation matrix L_{ij}^{α} appears in the coupling terms between the order parameter and the conserved variable, while the structure constants $f_{\alpha\beta\gamma}$ appears only in coupling terms among conserved variables. In the linear approximation of Q_i, the first term of Eq. (3.4) gives no contribution. To be concrete, we divide the matrix A into four blocks. The reversible coupling among order parameters may be ignored, so that $A_{ij}=0$. The coupling terms with conserved variables in the equations for the order parameters $A_{i\alpha}=iL_{ij}^{\alpha}Q_j$ are independent of q_α, so that $\partial A_{i\alpha}/\partial q_\alpha=0$. The two other blocks, $A_{\alpha i}=-iL_{ij}^{\alpha}Q_j$ and $A_{\alpha\beta}=-if_{\alpha\beta\gamma}q_\gamma$ appearing in the equations for the conserved variables also give zero contribution due to the antisymmetric character of the representation matrix $\hat{L}$ and the structure constants.

To get the term with derivatives in Eq. (3.4) we have to start from the nonlinear WT identities derived also in Sec. II D. In fact, by use of Eq. (2.104) we can repeat the derivation for the linear case and obtain the first term of Eq. (3.4). What we want to emphasize is that Kawasaki's formula (3.4) corresponds to the tree approximation in Eq. (2.103), so in principle we can go further. Another point is that the derivative terms, coming from the Jacobian, appear only for the basic fields, but not for the composite operators which also transform the derivative terms for the composite operators. Firstly, they may come from the loop corrections, secondly, and what is more likely in our opinion, they appear as a result of changes of measure in the path integral of the effective action (see the next section).

As a concrete example consider the simplest model of the isotropic antiferromagnet, i.e., model G in Ref. 10. This system consists of two densities, a nonconserved order parameter $\vec{Q}$ which is a three-component vector representing the staggered magnetization and a conserved density $\vec{q}$, also a three-component vector representing the total magnetization of the system.

From the commutation relations

$$[q_\alpha,q_\beta]=ig_0\epsilon_{\alpha\beta\gamma}q_\gamma \ , \ [Q_i,q_\alpha]=ig_0\epsilon_{i\alpha j}Q_j \ , \tag{4.30}$$

the structure constants and the representation matrix can be determined immediately

$$f_{\alpha\beta\gamma}=ig_0\epsilon_{\alpha\beta\gamma} \ , \ L_{ij}^{\alpha}=-ig_0\epsilon_{i\alpha j} \ , \tag{4.31}$$

where $\epsilon_{\alpha\beta\gamma}$ and $\epsilon_{i\alpha j}$ are fully antisymmetric unit tensors. Substituting Eq. (4.31) into Eqs. (4.23) and (4.29), and bringing together the external field term and the derivative of the free energy, i.e., changing $F\to F-J_iQ_i-J_\alpha Q_\alpha$, we obtain

$$\begin{aligned}\frac{\partial Q_i}{\partial \tau}&=-\sigma\frac{\delta F}{\delta Q_i}+g_0\epsilon_{\alpha ij}Q_j\frac{\delta F}{\delta q_\alpha} \ ,\\ \frac{\partial q_\alpha}{\partial \tau}&=l_{\alpha\beta}\nabla^2\frac{\delta F}{\delta q_\beta}-g_0\epsilon_{\alpha ij}\frac{\delta F}{\delta Q_i}Q_j-g_0\epsilon_{\alpha\beta\gamma}\frac{\delta F}{\delta q_\beta}q_\gamma \ .\end{aligned} \tag{4.32}$$

By taking $\sigma=\Gamma_0$, $l_{\alpha\beta}=\lambda_0\delta_{\alpha\beta}$ and changing to the vector notation, we retrieve the system of equations for model G.[10]

$$\begin{aligned}\frac{\partial \vec{Q}}{\partial \tau}&=-\Gamma_0\frac{\delta F}{\delta \vec{Q}}+g_0\vec{Q}\times\frac{\delta F}{\delta \vec{q}} \ ,\\ \frac{\partial \vec{q}}{\partial \tau}&=\lambda_0\nabla^2\frac{\delta F}{\delta \vec{q}}+g_0\vec{Q}\times\frac{\delta F}{\delta \vec{Q}}+g_0\vec{q}\times\frac{\delta F}{\delta \vec{q}} \ .\end{aligned} \tag{4.33}$$

The models A, B, and C are much simpler due to the absence of the reversible mode coupling. The other models such as E, F, H, and J models[10] and the SSS model[18] can be treated in the same way. We shall not repeat these simple calculations here.

V. LAGRANGIAN FORMULATION OF STATISTICAL FIELD THEORY

Suppose $\hat{\varphi}_i$, $i=1,2,\cdots l$, are the basic fields of the system, $\hat{Q}_i(\hat{\varphi}), i=1,2,\cdots n+m$, are composite operators representing the order parameters and conserved variables. Some of the basic fields may be order parameters also (as in the case of lasers). For simplicity we take all of them to be Hermitian Bose operators. In what follows operators will not be distinguished by special notations since their meaning is clear from the context.

Assuming the randomness of the initial phase, the density matrix is diagonal at the moment $\tau=\tau_0$:

$$\langle\varphi'(\vec{x},\tau_0)|\rho|\varphi''(\vec{x},\tau_0)\rangle = P(\varphi'(\vec{x}),\tau_0)\delta(\varphi'(\vec{x},\tau_0)-\varphi''(\vec{x},\tau_0)) \ . \tag{5.1}$$

The initial distribution of the macrovariables $Q_i(x)$ is given by

$$P(Q_i(\vec{x}),\tau_0) = \mathrm{tr}[\delta(Q_i(x)-Q_i(\varphi(x)))\rho] = \int[d\varphi(x)]\delta(Q_i(x)-Q_i(\varphi(x)))P(\varphi(\vec{x}),\tau_0) \ . \tag{5.2}$$

The generating functional for $Q_i(\phi(x))$ [Eq. (2.86)] under the assumption Eq. (5.1) can be written as

$$\begin{aligned} Z(J(x)) &= \exp[-iW(J(x))] \\ &= \mathrm{tr}\left\{T_p\left[\exp\left(-i\int J(x)Q(\varphi(x))\right)\right]\rho\right\} = N\int[d\varphi(x)]\exp\left(-i\int[\mathcal{L}(\varphi(x))-JQ(\varphi(x))]\right)\delta(\varphi_+-\varphi_-) \ , \end{aligned} \tag{5.3}$$

where

$$\delta(\varphi_+-\varphi_-) \equiv \int d\varphi'(x)\delta(\varphi(\vec{x},\tau_+=\tau_0)-\varphi'(\vec{x}))\delta(\varphi(\vec{x},\tau_-=\tau_0)-\varphi'(\vec{x}))P(\varphi'(\vec{x}),\tau_0) \ . \tag{5.4}$$

Multiplying the right-hand side of Eq. (5.3) by the normalization factor of the δ function on the closed time path

$$\int[dQ]\delta(Q_+-Q_-)\delta(Q(x)-Q(\varphi(x)))=1 \ , \tag{5.5}$$

changing the order of integration to replace $Q(\varphi(x))$ by $Q(x)$ and using the formula

$$\delta(Q(x)-Q(\varphi(x))) = \int\left[\frac{dI}{2\pi}\right]\exp\left(i\int_p[Q(x)-Q(\varphi(x))]I(x)\right) \ , \tag{5.6}$$

we can rewrite Eq. (5.3) as

$$Z(J) = N\int[dQ]\exp\left(iS_{\mathrm{eff}}(Q)-i\int_p JQ\right)\delta(Q_+-Q_-) \ , \tag{5.7}$$

where

$$e^{iS_{\mathrm{eff}}(Q)} \equiv \int\left[\frac{dI}{2\pi}\right]\exp\left(i\int_p QI-iW(I)\right) \ . \tag{5.8}$$

Here we are performing the direct and inverse Fourier transformations of the path integral. Since the continuous integration is taken over $I(x)$, $W(I)$ can be considered as the generating functional in the random external fields. Calculating the integral by Wentzel-Kramers-Brillouin (WKB) procedure in the one-loop approximation which is equivalent to the Gaussian averaging over the random fields, we obtain the effective action $S_{\mathrm{eff}}(Q)$ for macrovariables.

This is for the case when macrovariables are composite operators. The same is true, if all or part of macrovariables are basic fields themselves. A new field can also be introduced by using the δ function. Even if the initial distribution is multiplicative for different components

$$P(\varphi',\tau_0) = \prod_{i=1}^{n} P_i(\varphi'_i,\tau_0) \ ,$$

the Fourier transformations for the path integral have to be carried out simultaneously for all fields, since for the general case the Lagrangian of the system cannot be presented as a superposition of contributions from different components.

We now turn to discuss the general properties of the effective action $S_{\mathrm{eff}}(Q)$.

The generating functional for CTPGF's in the case of Hermitian Bose fields satisfies the relations

$$W(J_+(x),J_-(x))|_{J_+(x)=J_-(x)}=0 \ , \tag{5.9}$$

$$W^*(J_+(x),J_-(x)) = -W(J_-(x),J_+(x)) \ . \tag{5.10}$$

Taking successive functional derivatives of Eq. (5.9) and putting $J_+(x)=J_-(x)$ we get a number of relations between CTPGF's. It is easy to show by use of Eqs. (5.8) and (5.10) that

$$S^*_{\mathrm{eff}}(Q_+(x),Q_-(x)) = -S_{\mathrm{eff}}(Q_-(x),Q_+(x)) \ , \tag{5.11}$$

so S_{eff} is purely imaginary for $Q_+(x)=Q_-(x)$. Putting $Q_\pm(x) = Q + \Delta Q_\pm$ and taking functional expansion of Eq. (5.11) around Q, we obtain relations among functional derivatives of different order at the point Q:

$$\frac{\delta S}{\delta Q_+(x)} = \left[\frac{\delta S}{\delta Q_-(x)}\right]^* , \tag{5.12}$$

$$S_{Fij}(x,y) = S_{Fji}(y,x) = -S^*_{\tilde{F}ji}(y,x) ,$$
$$S_{\pm ij}(x,y) = S_{\mp ji}(y,x) = -S^*_{\mp ji}(y,x) , \tag{5.13}$$

where

$$S_{Fij}(x,y) \equiv \frac{\delta^2 S}{\delta Q_{i+}(x)\delta Q_{j+}(y)} , \tag{5.14}$$

etc.

If the system is invariant under the symmetry group G, i.e., the Lagrangian and the initial distribution do not change under

$$\varphi_i(x) \to \varphi_i^g(x) = U_{ij}(g)\varphi_j(x) ,$$
$$Q_i(\varphi) \to Q_i^g(\varphi) = V_{ij}(g)Q_j(\varphi) ,$$

then

$$W(J^g(x)) = W(J(x)), \quad J_i^g = J_i(x)V_{ji}^\dagger(g) ,$$
$$S_{\text{eff}}(Q^g(x)) = S_{\text{eff}}(Q(x)), \quad Q_i^g(\varphi) = V_{ij}(g)Q_j(\varphi) .$$

This is true if S_{eff} is calculated exactly. In fact, the symmetry properties of S_{eff}, although related to that of the original Lagrangian, may be different from the latter due to the averaging procedure.

If the lowest order of WKB, i.e., the tree approximation is taken in Eq. (5.8), it follows that

$$Q = \delta W/\delta I , \tag{5.15}$$

$$S_{\text{eff}}(Q) = -\Gamma(Q) . \tag{5.16}$$

In this case S_{eff} inherits all the properties of the generating functional $\Gamma[Q]$ for the vertex CTPGF's, i.e.,

$$S_{\text{eff}}(Q,Q) = 0 , \tag{5.17}$$

$$\delta S_{\text{eff}}/\delta Q_+|_{Q_+=Q_-=Q} = \delta S_{\text{eff}}/\delta Q_-|_{Q_+=Q_-=Q} , \tag{5.18}$$

$$S_F + S_{\tilde{F}} = S_+ + S_- , \tag{5.19}$$

$$\frac{\delta^l S_{\text{eff}}}{\delta Q_{i1}(1) \cdots \delta Q_{il}(l)} = i^{l-1}\langle T_p[Q_{i1}(1) \cdots Q_{il}(l)]_{1\text{PI}} , \tag{5.20}$$

where 1PI means one particle irreducible. According to Eqs. (5.16) and (2.78),

$$-iS_\pm(k) > 0 \tag{5.21}$$

after the Fourier transformation.

Near thermal equilibrium we have from Eq. (2.77)

$$S_{-ij}(k) - S_{+ij}(k) \underset{k_0\to 0}{\longrightarrow} -\beta k_0 S_{-ij}(k) . \tag{5.22}$$

Up to now we have discussed only the general properties of the effective action $S_{\text{eff}}(Q)$. In principle this can be derived from the microscopic generating functional W by averaging over the random external fields; it can also be constructed phenomenologically in accordance with the required symmetry properties. We shall now show that in the one-loop approximation in the path integral over dI and to the second order macrovariable fluctuations on positive and negative time branches, the current formulation of MSR field theory[13,14] is retrieved.

To calculate the integral (5.8) we expand the exponential factor around the saddle point, given by Eq. (5.15)

$$E \equiv \int_p QI - W = -\Gamma - \tfrac{1}{2}\int_p \Delta I W^{(2)} \Delta I + \cdots . \tag{5.23}$$

According to the computation rule described in Sec. II B, E can be rewritten as

$$E = -\Gamma - \tfrac{1}{2}\int \Delta\hat{I}^T \sigma_3 \hat{W}^{(2)} \sigma_3 \Delta\hat{I} , \tag{5.24}$$

where

$$\hat{W}^{(2)} = \begin{bmatrix} W_{++} & W_{+-} \\ W_{-+} & W_{--} \end{bmatrix}, \quad \Delta\hat{I} = \begin{bmatrix} \Delta I_+ \\ \Delta I_- \end{bmatrix} . \tag{5.25}$$

The result of the Gaussian integration, accurate to a constant multiplier is

$$e^{iS_{\text{eff}}(Q)} = e^{-i\Gamma(Q)}|\det(\sigma_3 \hat{W}^{(2)} \sigma_3)|^{-1/2} . \tag{5.26}$$

From the Dyson equation (2.59) we have

$$iS_{\text{eff}}[Q] = -i\Gamma(Q) + \tfrac{1}{2}\operatorname{tr}\ln\hat{\Gamma}^{(2)} , \tag{5.27}$$

where

$$\hat{\Gamma}^{(2)} = \begin{bmatrix} \Gamma_{++} & \Gamma_{+-} \\ \Gamma_{-+} & \Gamma_{--} \end{bmatrix}$$

is the two-point vertex function. By use of the transformation formula (2.33) we have

$$|\det\hat{\Gamma}^{(2)}| = |\det\tilde{\Gamma}^{(2)}| = |\det\Gamma_r||\det\Gamma_a| = |\det\Gamma_r|^2 , \tag{5.28}$$

where

$$\Gamma_r(x,y) = \delta^2\Gamma/\delta Q(x)\delta\Delta(y) , \tag{5.29}$$

$$Q(x) = \tfrac{1}{2}[Q_+(x) + Q_-(x)] ,$$
$$\Delta(x) = Q_+(x) - Q_-(x) . \tag{5.30}$$

As shown in Sec. IV

$$\left.\frac{\delta\Gamma}{\delta\Delta(y)}\right|_{\Delta=0} = \frac{1}{2}\left[\frac{\delta\Gamma}{\delta Q_+(y)} + \frac{\delta\Gamma}{\delta Q_-(y)}\right]_{\Delta=0}$$
$$= -\gamma\frac{\partial Q}{\partial t} - \frac{\delta F}{\delta Q(y)} . \tag{5.31}$$

It can be seen from comparing Eq. (5.31) with Eqs. (3.1) and (3.7) that $\delta^2\Gamma/\delta Q\,\delta\Delta$ is just the transformation matrix, accurate to the coefficient matrix γ, from ξ_i to Q_i, so the Jacobian can be calculated in the same way. Taking into account that the square power in Eq. (5.28) exactly cancels out the coefficient $\frac{1}{2}$ in Eq. (5.27) we have finally

$$iS_{\rm eff}(Q)=-i\Gamma(Q)-\frac{1}{2}\int\frac{dK}{\delta Q}dx \ , \qquad (5.32)$$

where

$$K=-\gamma^{-1}\delta F/\delta Q \ . \qquad (5.33)$$

In the path integral (5.7) the most plausible path is determined by the equations

$$\delta S_{\rm eff}(Q)/\delta Q_{\pm}=J_{\pm}(x) \ , \qquad (5.34)$$

$$Q(\vec{x},\tau_+=\tau_0)=Q(\vec{x},\tau_-=\tau_0) \ . \qquad (5.35)$$

Taking $J_+=J_-=J$, in the tree approximation of the path integral over $I(x)$, we obtain

$$\frac{\delta S_{\rm eff}(Q)}{\delta Q}=J(x)=\gamma\frac{\partial Q}{\partial t}+\frac{\delta F}{\delta Q} \ , \qquad (5.36)$$

which follows from Eqs. (4.6) and (5.16). This is just the TDGL equation.

We now consider the fluctuations around the most plausible trajectories. In the CTPGF approach, in addition to the fluctuations in the usual sense, field variables are permitted to take different values in positive and negative time branches. Changing variables in the path integral (5.7) to the usual time axis by introducing $Q(x)$ and $\Delta(x)$ according to Eq. (5.30) the effective action $S_{\rm eff}$ can be expanded as

$$S_{\rm eff}(Q_+(x),Q_-(x))=S_{\rm eff}(Q(x),Q(x))+\frac{1}{2}\int\left[\frac{\delta S_{\rm eff}}{\delta Q_+}+\frac{\delta S_{\rm eff}}{\delta Q_-}\right]\Delta(x)$$
$$+\frac{1}{8}\int\Delta(x)(S_{++}+S_{+-}+S_{-+}+S_{--})(x,y)\Delta(y)+\cdots \ .$$

Denoting

$$\frac{1}{4}i(S_{++}+S_{+-}+S_{-+}+S_{--})(x,y)\equiv-\gamma(x)\sigma(x,y)\gamma(y) \qquad (5.37)$$

and using Eqs. (5.17), (5.32), and (5.36) we obtain

$$e^{-iW(J(x))}=\int[dQ(x)][d\Delta(x)]\exp\left[-\frac{1}{2}\int\Delta(x)\gamma(x)\sigma(x,y)\gamma(y)\Delta(y)+i\int\gamma(x)\left[\frac{\partial Q}{\partial\tau}+\frac{\delta F}{\delta Q}\right]\Delta(x)\right.$$
$$\left.-\frac{1}{2}\int\frac{\delta K}{\delta Q}-i\int(J_\Delta Q+J_0\Delta)\delta(\Delta(\vec{x},\tau_0))\right] \ , \qquad (5.38)$$

where

$$J_\Delta=J_+(x)-J_-(x),\quad J_0=\frac{1}{2}[J_+(x)+J_-(x)] \ .$$

If we take $J_\Delta=J$ and change variables $\gamma(x)\Delta(x)\to\hat{Q}(x)$, $J_0\gamma^{-1}\to\hat{J}$ the generating functional for the MSR field theory [Eq. (3.13)] is retrieved. The Gaussian integration over $\Delta(x)$ gives

$$e^{-iW(J(x))}=N\int[dQ(x)]\exp\left\{-\frac{1}{2}\int\left[\frac{\partial Q(x)}{\partial\tau}+\frac{1}{\gamma}\left[\frac{\delta F}{\delta Q(x)}-\hat{J}(x)\right]\right]\sigma^{-1}(x,y)\right.$$
$$\left.\times\left[\frac{\partial Q(y)}{\partial\tau}+\frac{1}{\gamma}\left[\frac{\delta F}{\delta Q(y)}-\hat{J}(y)\right]\right]+\frac{1}{2}\int\frac{\delta}{\delta Q}\left[\frac{1}{\gamma}\frac{\delta F}{\delta Q}\right]-i\int JQ\right\} \ , \qquad (5.39)$$

which is the generating functional (3.14). It is interesting to point out that $\hat{J}=\frac{1}{2}(J_++J_-)$ corresponds to the physical external field, while $J=J_+-J_-$ corresponds to the formal source field used for construction of generating functional.

It can be seen by comparison of Eqs. (5.39) and (3.14) that $\sigma(x,y)$ presents the correlation matrix for random forces. If Q is a smooth function of x, σ can be taken as constant

$$\sigma=-(i/4\gamma^2)(S_F+S_{\tilde{F}}+S_++S_-)(k=0) \ . \qquad (5.40)$$

By use of Eq. (5.19), valid in the tree approximation, Eq. (5.40) can be rewritten as

$$\sigma = -(i/2\gamma^2)(S_+ + S_-) \ . \tag{5.41}$$

According to the definition of γ [Eq. (4.5)], i.e.,

$$\gamma = \lim_{k_0 \to 0} i(\partial/\partial k_0)\Gamma_r \ ,$$

taking into account that only the dissipative part gives any contribution, and using Eqs. (5.16) and (5.22) which are valid near the thermal equilibrium state, we obtain

$$\gamma = \lim_{k_0 \to 0} \frac{\partial}{\partial k_0} A = \frac{i}{2} \lim_{k_0 \to 0} \frac{\partial}{\partial k_0}(\Gamma_- - \Gamma_+)$$

$$= \tfrac{1}{2} i\beta\Gamma_- = -\tfrac{1}{4} i\beta(S_+ + S_-) \ . \tag{5.42}$$

Comparing Eq. (5.42) with Eq. (5.41) yields the fluctuation dissipation theorem

$$\sigma = 2/\beta\gamma \ ,$$

which in the ordinary notation is given by

$$\langle \xi(\tau)\xi(\tau')\rangle = 2\Gamma_0 kT\delta(\tau - \tau'), \quad \Gamma_0 = 1/\gamma \ . \tag{5.43}$$

For simplicity we derive here the generating functional for the single component Q. Extension to the multicomponent case is obvious.

VI. DISCUSSIONS

Summarizing the main results of this paper, we come to the following conclusions:

(i) The CTPGF approach is a natural theoretical framework for statistical field theory to describe systems with dominating long-wavelength fluctuations such as dynamical critical phenomena. By use of the CTPGF's the generalized Langevin equations for order parameters and conserved variables with mode coupling terms included in a natural way and the Lagrangian formulation of the classical field theory are deduced from a unified point of view. The perturbation theory of CTPGF in terms of $\hat{G}$ functions has the same structure as that for the ordinary field theory so it is simpler to deal with. In the current theory of critical dynamics and MSR field theory the perturbation expansion is constructed in terms of $\tilde{G}$ functions with two different types of propagators, i.e., the retarded and correlation functions. Therefore, the structure of such perturbation theory is more complicated. Another advantage of the CTPGF formalism is that the causality is guaranteed automatically. It does not need to be verified order by order, as in the existing theory.[19]

(ii) The noncommutativity of field operators, not obvious in the path integral formulation, is not a mathematical trick, but a necessity to describe the time evolution of the statistical field theory. Even if the infrared divergence of the terms, coming from the noncommutativity of operators, is lower than that for other functions, these terms are still needed when considering the time-dependent phenomena, since the infrared divergence of the response function is weaker than that for the correlation function (see Appendix A).

(iii) It can be seen from the calculations in this paper what kind of approximations are assumed in the existing theory of critical dynamics and what possible ways may be used to improve the current theory.

(a) In the existing theory the transport coefficient matrix for the coupling terms with conserved variables is assumed to be antisymmetric; i.e., only canonical motion is considered. It is possible to analyze the crossover effects of dissipative and canonical motions which may occur, in principle, in the framework of CTPGF's.

(b) The one-loop approximation in the path integral over the random fields corresponds to the Gaussian averaging. It is possible to go beyond the Gaussian approximation by calculating higher-loop corrections in the framework of CTPGF's.

(c) The current theory of critical dynamics corresponds to the second-order approximation of $\Delta(x)$, the fluctuations on positive and negative time branches. In principle higher-order corrections can be calculated. It may be more convenient to calculate directly the path integral for Q_+ and Q_-, not introducing $\Delta(x)$ explicitly.

(iv) The renormalization of the existing Lagrangian field theory is quite complicated.[13,14] One of the causes of such complexity lies in the fact that the number of vertices and primitive divergences is much greater than the number of coupling constants and also that Q and $\hat{Q}$ have different dimensions. It seems that the renormalization procedure will be simpler in terms of $\hat{G}$ functions, since different components of Green's function matrix have the same infrared divergence.

ACKNOWLEDGMENTS

One of the authors (L.Y.) would like to express his sincere gratitude to Professor H. Ehrenreich, Professor B. I. Halperin, Professor P. C. Martin, Professor D. Nelson, and Professor T. T. Wu for the kind hospitality they have extended to him at Harvard and for helpful discussions with them. He is also indebted for Professor A. Aharony, Dr. R. Bruinsma, Dr. B. L. Hu, Professor A. Jaffe, Dr. R. Morf, Dr. H. Sampolinsky, Professor M. Stephen, and Dr. A. Zippelius for interesting discussions. He is supported in part by the NSF through Grant No. DMR-77-10210.

APPENDIX A: RENORMALIZATION OF THE FINITE-TEMPERATURE FIELD THEORY

As shown by Zhou and Su[6] for the general case, the counter terms introduced in quantum field theory for $T=0$ K are enough to remove all ultraviolet divergences for the CTPGF's at any temperature. Other authors (see references cited in Ref. 6) come to the same conclusion for finite temperature field theory without resorting to the CTPGF formalism. This result is reasonable from the physical point of view since the statistical average does not change the properties of systems at very short distances and therefore does not contribute new ultraviolet divergences. What we should like to point out is that in considering the phase transitionlike phenomena it is necessary to separate first the leading infrared divergent term and then to carry out the ultraviolet renormalization which is different from that for the usual quantum field theory.

To be concrete, consider the relativistic scalar Bose field, the CTPGF propagators for which can be written as[4]

$$G_{++}(k) \equiv \Delta_F(k) = \frac{1}{k^2-m^2+i\epsilon} - 2\pi i n(\vec{k})\delta(k^2-m^2) \ ,$$

$$G_{-+}(k) \equiv \Delta_-(k) = -2\pi i \delta(k^2-m^2)[\Theta(k_0)+n(\vec{k})] \ ,$$

$$G_{+-}(k) \equiv \Delta_+(k) = -2\pi i \delta(k^2-m^2)[\Theta(-k_0)+n(\vec{k})] \ ,$$

$$G_{--}(k) \equiv \Delta_{\bar{F}}(k) = \frac{-1}{k^2-m^2-i\epsilon} - 2\pi i n(\vec{k})\delta(k^2-m^2) \ , \tag{A1}$$

where

$$n(\vec{k}) = \{\exp[\epsilon(\vec{k})/T]-1\}^{-1}, \quad \epsilon(\vec{k}) = (\vec{k}^2+m^2)^{1/2} \ . \tag{A2}$$

Near the phase transition point $m \approx 0$.

$$\epsilon(\vec{k})/T << 1, \quad n(\vec{k}) \approx T/\epsilon(\vec{k}) >> 1 \tag{A3}$$

for the long-wavelength excitations. Since the $n(\vec{k})$ terms appear together with the δ function, i.e., on the mass shell, the integration over frequencies can be carried out automatically, so the infrared divergence of these terms is higher than that for other terms by one order of magnitude. Therefore the marginal space dimension for renormalizability for finite temperature φ^4 theory is $d_c=4$, not $d_c=4-1$ as in the case of the ordinary field theory. This is what is meant by saying "quantum system in d dimensions corresponds to the classical system in $d+1$ dimensions."

What has been said above can be verified explicitly by calculating the primitive divergent diagrams for mass, vertex, and wave function renormalization, carrying out the frequency integration, and taking the high-temperature limit $T >> \epsilon(\vec{k})$ to retrieve the results which are identical with that of the current theory of critical phenomena.[20] It is much easier to verify this by use of Matsubara Green's functions, retaining only terms $\omega_n=0$ in the frequency summation.

Some investigators of finite-temperature field theory improperly use the renormalization constants for the $T=0$ K case to study phase transition related phenomena. Since the high-temperature limit has been taken for the case of phase transition both relativistic and quantum effects are unimportant. The only possible exception is phase transition near $T=0$ K, where both statistical and quantum fluctuations play their parts. As far as phase transition is concerned the ordinary field models cannot give anything new beyond the current theory of critical phenomena. (The situation for the nonabelian gauge models is not quite clear.)

The noncommutativity of operators is not essential for the static phenomena, that implies the four propagators in Eq. (A1) may be replaced by the correlation function $-2\pi i n(\vec{k})\delta^2(K^2-m^2)$. This is not the case for dynamic phenomena. The first term of G_{++} and G_{--} comes from the inhomogeneous term of Green's function equation, i.e., the commutator. If only the leading infrared divergent terms are retained the four propagators become equal to one another, so that the retarded function

$$G_r = G_{++} - G_{+-} = 0 \ .$$

Therefore the retarded Green's functions are less infrared divergent than the correlation functions. To treat them properly, the noncommutativity of operators has to be taken into account, even though this is a "purely" classical field theory. It is easy to show that all these properties illustrated with the free propagators remain true for the renormalized propagators.

As mentioned in the Introduction, the high-temperature limit of statistical field theory corresponds to the "super Bose" limit, but not the Boltzmann limit. Usually we consider the classical fields to be commutative, since unity can be ignored in comparison with n which is large. If unity cannot be neglected in the phenomena under study we have to start with the noncommutative operators. This is one of the reasons why statistical field theory and quantum field theory have so close an analogy. The physical implications of this analogy are discussed elsewhere.[21]

APPENDIX B: FURTHER RESULTS ON TRANSFORMATIONS OF CTPGF

The main results concerning the transformations of different forms for CTPGF's are described in Sec. II B. Here we shall prove the theorems (2.37) and (2.40), illustrate them by more complicated examples, and discuss the transformations for connected Green's functions.

It is more convenient for some cases to introduce the spinor notation. Let

$$Q \equiv \begin{pmatrix} \eta^T \\ \xi^T \end{pmatrix}, \quad Q^T \equiv (\eta, \xi) \ ; \tag{B1}$$

i.e.,

$$Q_1 = \eta^T = (1/\sqrt{2})(1,-1) \ , \quad Q_2 = \xi^T = (1/\sqrt{2})(1,1) \ ,$$

$$Q_1^T = \eta = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ , \quad Q_2^T = \xi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \ . \tag{B2}$$

The normalization condition

$$QQ^T = I$$

is expressed as

$$\eta^\alpha \eta_\alpha = \xi^\alpha \xi_\alpha = 1 \ , \quad \eta^\alpha \xi_\alpha = \xi^\alpha \eta_\alpha = 0 \ , \tag{B3}$$

where η^α, ξ^α are components of η^T and ξ^T, etc.

In spinor notation the transformation from $\hat{G}$ to $\tilde{G}$ [Eq. (2.35)] is

$$G_{2\cdots 2(k),1\cdots 1(n-k)}(12\cdots n) = 2^{n/2-1}\xi^{\alpha_1}\cdots \xi^{\alpha_k}\eta^{\alpha_{k+1}}\cdots \eta^{\alpha_n}G_{\alpha_1\cdots\alpha_n}(12\cdots n) \ , \tag{B4}$$

while the inverse transformation from $\tilde{G}$ to $\hat{G}$ [Eq. (2.36)] is

$$\begin{aligned} G_{\alpha_1\alpha_2\cdots\alpha_n}(12\cdots n) = 2^{1-n/2}(&\xi_{\alpha_1}\cdots\xi_{\alpha_n}G_{22\cdots 2} + \eta_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n}G_{12\cdots 2} + \xi_{\alpha_1}\eta_{\alpha_2}\cdots\xi_{\alpha_n}G_{212\cdots 2} + \cdots \\ &+ \xi_{\alpha_1}\cdots\xi_{\alpha_{n-1}}\eta_{\alpha_n}G_{2\cdots 21} + \cdots + \eta_{\alpha_1}\cdots\eta_{\alpha_{n-1}}\xi_{\alpha_n}G_{1\cdots 12} \\ &+ \eta_{\alpha_1}\cdots\xi_{\alpha_{n-1}}\eta_{\alpha_n}G_{1\cdots 21} + \cdots + \xi_{\alpha_1}\eta_{\alpha_2}\cdots\eta_{\alpha_n}G_{21\cdots 1}) \ . \end{aligned} \tag{B5}$$

The generating functional $Z(J(x))$ can be expanded as

$$\begin{aligned} Z(J(x)) &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_p \cdots \int_p \frac{\delta^n Z}{\delta J(1)\cdots\delta J(n)}\bigg|_{J=0} J_p(1)\cdots J_p(n) \\ &= 1 - i\sum_{n=1}^{\infty}\frac{1}{n!}\int_p\cdots\int_p G_p(1\cdots n)J_p(1)\cdots J_p(n) \\ &= 1 - i\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}G_{\alpha_1\cdots\alpha_n}(1\cdots n)(\sigma_3\hat{J})_{\alpha_1}\cdots(\sigma_3\hat{J})_{\alpha_n} \ . \end{aligned} \tag{B6}$$

If we take $J_+(x) = J_-(x) = J(x)$, Eq. (B6) becomes

$$Z(J(x)) = 1 - i\sum_{n=1}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}G_{\alpha_1\cdots\alpha_n}(1\cdots n)\eta^{\alpha_1}\cdots\eta^{\alpha_n}J(1)\cdots J(n) \ . \tag{B7}$$

According to the normalization condition (2.16)

$$Z(J(x))|_{J_+(x)=J_-(x)=J(x)} = 1$$

and considering the arbitrariness of $J(x)$ we obtain

$$\tilde{G}_{11\cdots 1}(12\cdots n) = 2^{n/2-1}\eta^{\alpha_1}\eta^{\alpha_2}\cdots\eta^{\alpha_n}G_{\alpha_1\alpha_2\cdots\alpha_n}(12\cdots n) = 0 \tag{B8}$$

thus the first theorem (2.37) has been proved.

It follows from Eq. (B8) for three-point functions that

$$G_{+++} + G_{+--} + G_{-+-} + G_{--+} = G_{---} + G_{++-} + G_{+-+} + G_{-++} \ . \tag{B9}$$

Similarly, for four-point functions we have

$$G^{(+)} \equiv \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(1+\alpha\beta\gamma\delta)G_{\alpha\beta\gamma\delta} = G^{(-)} \equiv \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(1-\alpha\beta\gamma\delta)G_{\alpha\beta\gamma\delta} \ ; \tag{B10}$$

i.e, the sums of terms with the same signature are equal.

To prove the second theorem we first multiply Eq. (B4) by the normalization condition of step functions (2.29) to obtain

$$G_{2\cdots 2(k),1\cdots 1(n-k)}(1\ \cdots\ n)=(-i)^{n-1}2^{n/2-1}\sum_{p\begin{pmatrix}1\cdots n\\ p_1\cdots p_n\end{pmatrix}}\Theta(p_1\ \cdots\ p_n)\xi^{\alpha_1}\ \cdots\ \xi^{\alpha_k}\eta^{\alpha_{k+1}}\ \cdots\ \eta^{\alpha_n}$$

$$\times\langle T_p(\varphi_{\alpha_1}(1)\ \cdots\ \varphi_{\alpha_n}(n))\rangle\ . \tag{B11}$$

Let

$$\zeta^{p_i}=\begin{cases}\xi^{p_i}, & \text{if } 1\le p_i\le k,\\ \eta^{p_i}, & \text{if } k+1\le p_i\le n,\end{cases} \tag{B12}$$

we have

$$\Theta(p_1\ \cdots\ p_n)\xi^{\alpha_1}\ \cdots\ \xi^{\alpha_k}\eta^{\alpha_{k+1}}\ \cdots\ \eta^{\alpha_n}\langle T_p(\varphi_{\alpha_1}(1)\ \cdots\ \varphi_{\alpha_n}(n))\rangle$$

$$=\Theta(p_1\ \cdots\ p_n)\zeta^{p_1}\ \cdots\ \zeta^{p_n}\langle T_p(\varphi_{p_1}(p_1)\ \cdots\ \varphi_{p_n}(p_n))\rangle \tag{B13}$$

because of the definition of the Θ function (2.26) and the symmetry of CTPGF.

Since the time ordering in the usual sense has already been fixed by the Θ function the action of the T_p operator is reduced to

$$\zeta^{p_n}\langle T_p(\varphi_{p_1}(p_1)\ \cdots\ \varphi_{p_n}(p_n))\rangle=\frac{1}{\sqrt{2}}\langle (T_p(\varphi_{p_1}(p_1)\ \cdots\ \varphi_{p_{n-1}}(p_{n-1})),\ \varphi(p_n))\rangle\ ,$$

where

$$(\ ,\ )=\begin{cases}[\ ,\ ], & \text{if } \zeta^{p_n}=\eta^{p_n}\ ,\\ \{\ ,\ \}, & \text{if } \zeta^{p_n}=\xi^{p_n}\ .\end{cases} \tag{B14}$$

Such a process of getting rid of T_p can be continued up to the last step to get zero, if $\zeta^{p_1}=\eta^{p_1}$ or

$$2^{-n/2+1}((\ \cdots\ (\varphi(p_i),\varphi(p_2))\ \cdots\ ),\varphi(p_n))\ ,$$

if $\zeta^{p_1}=\xi^{p_1}$. We see that the factor $2^{-n/2+1}$ exactly cancels out the numerical coefficient $2^{n/2-1}$, so proving the theorem (2.40). In Sec. II B we have considered the two-point functions. As a further example we have for $n=3$

$$G_{211}(123)=(-i)^2\sum_{p\in\begin{pmatrix}23\\ ij\end{pmatrix}}\Theta(1ij)\langle[[1,i],j]\rangle\ ,$$

$$G_{221}(123)=(-i)^2\sum_{p\in\begin{pmatrix}12\\ ij\end{pmatrix}}(\Theta(ij3)\langle[\{i,j\},3]\rangle+\Theta(i3j)\langle\{[i,3],j\}\rangle)\ , \tag{B15}$$

$$G_{222}(123)=(-i)^2\sum_{p\in P_3}\Theta(ijk)\langle\{\{i,j\},k\}\rangle=(-i)^2\langle\{\{1,2\},3\}\rangle\ ,$$

and for $n=4$

$$G_{2111}(1234)=(-i)^3\sum_{p\in\begin{pmatrix}234\\ ijk\end{pmatrix}}\Theta(1ijk)\langle[[[1,i],j],k]\rangle\ ,$$

$$G_{2211}(1234)=(-i)^3\sum_{p\in\begin{pmatrix}12\\ ij\end{pmatrix}\begin{pmatrix}34\\ kl\end{pmatrix}}(\Theta(ijkl)\langle[[\{i,j\},k],l]\rangle+\Theta(ikjl)\langle[\{[i,k],j\},l]\rangle+\Theta(iklj)\langle\{[[i,k],l],j\}\rangle)\ , \tag{B16}$$

$$G_{2221}(1234)=(-i)^3\sum_{p\in\begin{pmatrix}123\\ ijk\end{pmatrix}}(\Theta(ijk4)\langle[\{\{i,j\},k\},4]\rangle+\Theta(ij4k)\langle\{[\{i,j\},4],k\}\rangle+\Theta(i4jk)\langle\{\{[i,4],j\},k\}\rangle)\ ,$$

$$G_{2222}(1234)=(-i)^3\langle\{\{\{1,2\},3\},4\}\rangle\ ,$$

where for short we write $i \equiv \varphi(i)$, etc.

It is interesting to note that although all the possible combinations of retarded, advanced, and correlation functions are realized in real time, they are defined by the CTPGF approach in a quite natural way.

The first theorem (2.37) is valid also for the connected CTPGF, since we can repeat word for word the proof starting from Eq. (2.17). This is not the case for the second theorem, where some complications appear. It follows from the definitions of disconnected [Eq. (2.4)] and connected [Eq. (2.15)] CTPGF's that

$$G_p^c(1) = G_p(1) \ , \tag{B17a}$$

$$G_p^c(1,2) = G_p(1,2) + iG_p(1)G_p(2) \ , \tag{B17b}$$

$$G_p^c(1,2,3) = G_p(1,2,3) + i[G_p(1)G_p(2,3) + G_p(2)G_p(1,3) + G_p(3)G_p(1,2)] - 2G_p(1)G_p(2)G_p(3) \ldots \ . \tag{B17c}$$

It can be shown by use of Eq. (B17) that the formulas for all "purely" retarded functions remain true, as for example

$$G_{21}^c(1,2) = G_{21}(1,2) = -i\Theta(1,2)\langle[1,2]\rangle \ , \quad G_{211}^c(1,2,3) = G_{211}(1,2,3) \ , \quad G_{2111}^c(1,2,3,4) = G_{2111}(1,2,3,4) \ldots \ . \tag{B18}$$

These functions are similar to r functions used to construct the Lehmann-Symanzik-Zimmermann (LSZ)[22] axiomatic field theory, which are the same for both connected and disconnected functions. All other functions are modified, for example,

$$G_c^c(12) = -i(\langle\{1,2\}\rangle - \{\langle 1\rangle, \langle 2\rangle\}) \ , \tag{B19}$$

$$\begin{aligned} G_{122}^c(123) = (-i)^2 \sum_{p_2} \{ & \Theta(p_2,p_3,1)(\langle[\{p_2,p_3\},1]\rangle - \langle[\langle p_2\rangle p_3,1]\rangle - \langle[p_2\langle p_3\rangle,1]\rangle) \\ & + \Theta(p_2,1,p_3)(\langle\{[p_2,1],p_3\}\rangle - \langle[p_2,1]\rangle\langle p_3\rangle)\} \ , \end{aligned} \tag{B20}$$

$$G_{222}^c(123) = (-i)^2 \sum_{p_3} \Theta(p_1p_2p_3)(\langle\{\{p_1,p_2\},p_3\}\rangle - \langle\{p_1,p_2\}\rangle\langle p_3\rangle + 2\{\{\langle p_1\rangle, \langle p_2\rangle\}, \langle p_3\rangle\}) \ , \tag{B21}$$

$$G_{2211}^c(1234) = G_{2211} + 2i[G(1)G_{211}(234) + G(2)G_{211}(134)] + 2i[G_r(13)G_r(24) + G_r(14)G_r(23)] \ , \tag{B22}$$

$$\begin{aligned} G_{2221}^c(1234) = {} & G_{2221}(1234) + 2i[G(1)G_{221}(234) + G(2)G_{221}(134) + G(3)G_{221}(124)] \\ & + 2i[G_c(12)G_r(34) + G_c(13)G_r(24) + G_c(23)G_r(14)] \\ & - 8[G(1)G(2)G_r(34) + G(1)G(3)G_r(24) + G(2)G(3)G_r(14)] \ . \end{aligned} \tag{B23}$$

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*On leave from the Institute of Theoretical Physics, Academia Sinica, Beijing, China.

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