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Applied Symbolic Dynamics*

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Symbolic dynamics is a coarse-grained description of dynamics. By taking into account the "geometry" of the dynamics, it can be cast into a powerful tool for practitioners in nonlinear science. Detailed symbolic dynamics can be developed not only for one-dimensional mappings, unimodal as well as those with multiple critical points and discontinuities, but also for some two-dimensional mappings. The latter paves the way for a symbolic dynamics study of ordinary differential equations via the Poincaré maps. This paper provides an overview of the recent development of the applied aspects of symbolic dynamics.

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I. Introduction

Symbolic dynamics is a rigorous way to study complex dynamics with *finite* precision. As an abstract chapter in the mathematical theory of dynamical systems [1,2], it originated from the work of Hadamard [3] and Morse [4]. The basic idea is very simple: divide the phase space into a finite number of regions and label each region by a letter from a certain alphabet; instead of following a trajectory point by point one only keeps recording the alternation of letters. One loses a great amount of detailed information on the dynamics, but some essential, robust, features of the motion may be kept, e.g., periodicity or chaoticity of an orbit. This is nothing but what physicists call a coarse-grained description.

The idea of symbolic dynamics applies to dynamics in any finite-dimensional phase space. In many cases, say, for theorem-proving, an arbitrary partition of the phase space would do the job. However, only for one-dimensional mappings symbolic dynamics has been developed more or less completely. This is due to the nice ordering property of real numbers on an interval and due to the possibility of partitioning the "phase space", i.e., the interval, in accordance with the "geometry" of the dynamics. In fact, many useful rules and beautiful results have been derived. Recently, significant progress has been made in symbolic dynamics of two-dimensional maps, but the achievement is still rather limited compared to what has been known in one dimension. Nevertheless, the knowledge of symbolic dynamics in one and two dimensions proves to be quite instructive in understanding the systematics of periodic orbits and chaotic behavior in some higher-dimensional dissipative systems, e.g., the Lorenz model and some periodically forced nonlinear oscillators. The presence of dissipation is essential, since it causes the shrinking of phase space volume, which makes a "strange attractor" closer to a low-dimensional objects at least in certain sections of the attractor.

Chaotic dynamics of dissipative systems provides a rare and lucky case in physics, when low-dimensional systems are not merely toy models, but lead to essential "universal" results which are quite useful in understanding higher dimensional systems. In a sense, everyone who enters the field of chaos should start with the study of symbolic dynamics. We have called this approach applied symbolic dyrlamics [5-8].

Applied symbolic dynamics commenced from a seminal paper by Metropolis, Stein and Stein [9]. The kneading theory of Milnor and Thurston [10], the lecture of Guckenheimer [11], and a paper by Derrida, Gervois, and Pomeau [12], among others, further developed the theory. What had been known by the end of 1970s was summarized in the book by Collet and Eckmann [13]. There has been significant generalization and simplification of the theory both for one-dimensional and two-dimensional mappings as well as their application to ordinary differential equations since the mid 1980s, for details see, e.g., [7, 8].

II. One-dimensional mappings

We consider one-dimensional maps of the general form

$$x_{n+1} = f(\mu, x_n),$$

where $f(\mu, x)$ is a nonlinear "mapping function" of the variable x and μ is a set of parameters. The function $f(x)$ maps an interval I into itself; it may have several monotone pieces between "turning" points and discontinuities. Symbolic dynamics of such maps has been understood more or less completely. We summarize some main points:

1. The phase space, i.e., the interval, is partitioned according to the monotone branches of the mapping function. Any numerical orbit corresponds to a semi-infinite symbolic sequence and a functional composition represented by the same set of symbols, understood as inverse functions of the monotone branches.

2. All symbolic sequences for a given type of maps may be ordered. Admissibility conditions based on ordering rules may be formulated to test whether a given symbolic sequence is reproducible in the dynamics or not.

3. There is a Periodic Window Theorem: any admissible superstable periodic sequence may be extended to a "window" with its upper and lower sequences. It leads to a method of generating the shortest admissible superstable periodic sequence in between any two given admissible periodic sequences.

4. There is a word-lifting technique [14, 15] which allows one to determine the parameter of any given type of superstable periodic and eventually periodic orbit.

5. There are composition rules which generate more admissible sequences from known ones. The simplest rule is called the *-composition [12] and it has a close relation with possible fine structure in the power spectra of observed periodic orbits.

6. The counting problem on how many periodic orbits exist for a given map has been solved completely for continuous maps [16] and partly for maps with discontinuity [17].

7. Topological entropy of superstable periodic and eventually periodic sequences may be calculated from transfer matrices which may be written down directly from the symbolic

sequences without knowing the precise form of the mapping function.

8. Maps with multiple critical points and discontinuities are best parameterized by their kneading sequences. The parameter space, called also the kneading space, may be constructed by using the admissibility conditions.

9. Circle maps, i.e., maps from a circle to itself, though may be studied as that with multiple critical points and discontinuities, do possess some specific features dictated by the topology of the phase space. Their study is facilitated by the Farey representation of rational numbers and the associated machinery, see, e.g., Chapter 4 of [8].

10. For maps with a discrete symmetry there are the phenomena of symmetry breaking and restoration which may be analyzed by using symbolic dynamics [18].

11. Symbolic sequences in the unimodal maps are naturally related to formal language and the theory of grammatical complexity. Periodic and eventually periodic sequences are the only types of regular language. The transfer matrix provides a way to go beyond regular languages [19]. There are examples of context-dependent languages of different complexity but no known example of context-free language yet. Hence a conjecture: no context-free language exists in the languages associated with unimodal maps. A good reference to this set of problems is [20].

12. Periodic orbits in unimodal maps are related to knots in 3-space. There are some observations but not much rigorous results, see, e.g., Chapter 9 in [8].

III. **Two-dimensional mappings**

In two- and higher-dimensional systems the nice ordering property of real numbers and the simple partition of an interval, which have played crucial role in symbolic dynamics of one-dimensional maps, no longer exist. In addition, 2D maps usually lead to bi-infinite symbolic sequences. The partition for the Hénon map [21], using tangencies of the invariant manifolds of the fixed points, was first discussed by Grassberger and Kantz [22] in order to calculate its topological entropy. Then Cvitanović, Gunaratne, and Procaccia [23] used the partition to develop symbolic dynamics. Later on the role of forward and backward foliations of the map in determining the partition lines has been recognized by Zheng and collaborators. In fact, the generalization from tangent points between the stable and unstable manifolds to that between the two dynamical foliations are essential and necessary, as it was shown analytically on the example of two piecewise linear maps. The simplest case turns out to be the two-dimensional version of the sawtooth map, introduced by Tel [24]. Its symbolic dynamics was constructed in [25]. The piecewise linear counterpart of the Hénon map, socalled Lozi map [26], may be treated in a similar manner [27, 28]. The two piecewise linear maps helped to reach a deeper understanding of the symbolic dynamics of the Hénon map [29-32]. However, the symbolic dynamics of Hénon map has not been understood thoroughly on the whole parameter plane.

We mention in passing that dynamical foliations may be constructed for Hamiltonian systems as well especially when the Poincaré sections may be reduced to two-dimensional, hence leading to a better symbolic dynamics, see [33, 34].

VI. Ordinary differential equations

In order to cut a long story short we will only mention a few results of applying symbolic dynamics to ordinary differential equations.

Some years ago we have applied symbolic dynamics of one-dimensional maps to the systematics of periodic orbits in differential equations. In particular, the ordering of periodic orbits of the periodically forced Brusselator [35] has been compared to that of the quadratic map, using symbolic dynamics of two letters [36]. The systematics of periodic orbits [37] in the autonomous Lorenz model has been juxtaposed with the ordering of kneading sequences in the antisymmetric cubic map with and without a discontinuity [38]. These essentially one-dimensional studies were summarized in [39].

Our main argument for using 1D maps lies in the shrinking of phase space volume due to dissipation. However, the Poincaré maps of ordinary differential equations are necessarily two-dimensional and there is no a priori reason that the two-dimensional nature will not show off. Having reached a better understanding of symbolic dynamics of two-dimensional maps, we have undertaken the job of justifying the previous one-dimensional approach and revealing the cases where a two-dimensional study leads to essentially new insight. We list some recent references: the periodically forced Brusselator [40,41], the forced two-well Duffing equation [42], the NMR-laser model [43,44], and the Lorenz model [45-48]. In the Lorenz equations numerical work under the guidance of topology, i.e., symbolic dynamics, has yield all stable and unstable periodic orbits up to period 6 in a wide parameter range [48].

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